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INVERSION FORMULA FOR THE CONTINUOUS LAGUERRE WAVELET TRANSFORM

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Abstract

In this paper, we accomplished the concept of convolution of Laguerre transform for the study of continuous Laguerre wavelet transform and discuss some of its basic properties. Finally our main goal is to find out the Plancherel and inversion formula for the Continuous Laguerre Wavelet Transform.

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I. Introduction

The wavelet transform for a function $f \in L^2(R)$ with respect to the wavelet $\phi \in L^2(R)$ Is defined by

$$(W_\varphi f)(b,a) = \int_{-\infty}^{\infty} f_1(t)\overline{\varphi_{b,a}(t)}dt, b \in R_+, a > 0 \qquad (1.1)$$

Where,

$$\varphi_{b,a}(t) = a^{-1/2}\varphi\left(\frac{t-b}{a}\right). \qquad (1.2)$$

Translation τ_b is defined by

$$\tau_b\varphi(t) = \varphi(t-b), b \in R_+$$

And dilation D_a is defined by

$$D_a\varphi(t) = a^{-1/2}\phi\left(\frac{t}{a}\right), a > 0,$$

We can write

$$\phi_{b,a}(t) = \tau_b D_a \phi(t). \tag{1.3}$$

From (1.1), (1.2) and (1.3) it is obvious that wavelet transform of the function f on R is an integral transform for which the kernel is the dilated translate of ϕ

We can also express (1.1) as the convolution:

$$(W_\varphi f_1)(b,a) = (f_1 * f_{2,a})(b), \tag{1.4}$$

Where,

$$f_2(t) = \overline{\varphi(-t)}.$$

As for every integral transform there exists a particular type of convolution, one can define wavelet transform with respect to a integral transform using associated convolution. Integral transform including special functions as kernel play a significant role in the theory of partial differential equations.

Pathak and Dixit [VII] have defined Bessel wavelet using Bessel functions on semi-infinite interval $(0,\infty)$. Then in [VIII] Upadhyay and Thripathi construct continuous wavelet transform corresponding to Watson transform. Motivating from above ideas we are interested to define the wavelet transform corresponding to Laguerre transform and to establish inversion and Plancherel formula for Laguerre wavelet transform.

II. Preliminaries

Let

$$\phi_{\lambda,m}(x,t) = e^{i\lambda t} e^{-\frac{|\lambda| x^2}{2}} \frac{L_m^{(\alpha)}\left(|\lambda| x^2\right)}{L_m^{(\alpha)}(0)}, \ (x,t) \in [0,\infty) \text{x } \mathbf{R}, \tag{2.1}$$

Where $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α , a non-negative real number.

Let

$$\mathscr{L}_n^\alpha(x) = \frac{n!\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} L_n^\alpha(x), \quad x \in [0,\infty), \tag{2.2}$$

Where $L_n^\alpha(x)$ is the Laguerre polynomial of degree n and order α >-1 is a kernel.

Set

$$d\wedge(x)=\frac{1}{\Gamma(\alpha+1)}e^{-x}\ \ x^\alpha dx, \tag{2.3}$$

And

$$\mathscr{L}_n^\alpha(x)=\rho(n)\Gamma(\alpha+1)L_n^\alpha(x),\ \rho(n)=\frac{n!}{\Gamma(n+\alpha+1),} \tag{2.4}$$

Let f(x) be a measurable function defined on $[0,\infty)$. Then its Laguerre transform $\hat{f}(n)$[2] is given by

$$\hat{f}(n)=\int_0^\infty f(x)\ \mathscr{L}_n^\alpha(x)\ d\wedge(x). \tag{2.5}$$

The inverse Laguerre transform is given by

$$\mathscr{L}[f]^v(x)=f(x)=\sum_n \hat{f}(n)\,\mathscr{L}_n^\alpha(x)\ \sigma(n), \tag{2.6}$$

Where $\sigma(n)=\dfrac{1}{\Gamma(\alpha+1)\rho(n)}$.

Let $L_{p,\wedge}[0,\infty),\ 1\le p\le\infty$, denote the space of those real measurable functions f on $[0,\infty)$ for which

$$\|f\|_{p,\wedge}=\left[\int_0^\infty |f(x)|^p\ d\wedge(x)\right]^{\frac{1}{p}}<\infty,\ 1\le p<\infty, \tag{2.7}$$

$$\|f\|_\infty=\underset{0\le x<\infty}{\text{ess}}\ \ \sup|f(x)|<\infty. \tag{2.8}$$

An inner product on $L_{2,\wedge}$, is given by

$$<f,g>_\wedge=\int_0^\infty f(x)\ \overline{g(x)}\ d\wedge(x).$$

For any $f\in L_{2,\wedge}$ the following Parseval identity holds for the Laguerre transform:

$$\sum_{n} \sigma(n)\,\hat{f}(n)\hat{g}(n) = \int_0^\infty f(x)g(x)\,d\wedge(x). \tag{2.9}$$

And

$$\sum_{m} \sigma(n)\,\hat{f}(n)\hat{g}(n) = \int_0^\infty L^{-1}\left[\hat{f}(n)\right]L^{-1}\left[\hat{g}(n)\right]d\wedge(x). \tag{2.10}$$

The Laguerre translation τ_y of $f \in L_{p,\wedge}[0,\infty), 1 \le p < \infty,$ is defined by

$$\tau_y f(x) = f(x,y) = \int_0^\infty f(z)d(x,y,z)\,d\wedge(z), 0 < x, y < \infty, \tag{2.11}$$

Where,

$$d(x,y,z) = \sum_{n=0}^{\infty} \mathscr{L}_n^\alpha(x)\mathscr{L}_n^\alpha(y)\mathscr{L}_n^\alpha(z)\sigma(n). \tag{2.12}$$

The Laguerre convolution [4] off $\in L_{p,\wedge}[0,\infty)$ and $g \in L_{q,\wedge}[0,\infty)$ is defined by

$$(f * g)(x) = \int_0^\infty \tau_y f(x)\, g(y)\, d\wedge(y). \tag{2.13}$$

Using relation (2.11), $(f * g)$ can also be expressed as

$$(f * g)(x) = \int_0^\infty\int_0^\infty f(z)g(y)d(x,y,z)\; d\wedge(y)d\wedge(z). \tag{2.14}$$

The convolution $(f * g)$ defined by (2.14) satisfies the following norm inequality:

(i) $$\| f * g \|_{r,\wedge} \le \| f \|_{p,\wedge} \| g \|_{q,\wedge}\, . \tag{2.15}$$

Moreover, for f, g $\in L_{1,\wedge}$, we have

(ii) $$(f * g)^\wedge(n) = \hat{f}(n)\,\hat{g}(n). \tag{2.16}$$

III. Continuous Laguerre Wavelet Transform

For a function $\phi \in L_{p,\wedge}[0,\infty)$, define the Laguerre wavelet $\psi_{b,a}(t)$ is as follows:

$$\begin{aligned}\phi_{b,a}(t) &= \tau_b D_a \phi(t)\\ &= \tau_b\ \phi(at)\end{aligned} \tag{3.1}$$

Where τ_b translation operator associated with Laguerre transform and D_a is dilation operator.

III.i. Admissible Laguerre Wavelet

The function $\phi \in L_{2,\wedge}[0,\infty)$ is called admissible Laguerre wavelet if ϕ satisfies the below admissibility condition:

$$c_\phi = \sum_n \frac{\left|\hat{\phi}(n)\right|^2}{n} < \infty \tag{3.2}$$

III.ii. Continuous Laguerre Wavelet Transform

For $\phi \in L_{p,\wedge}$ we describe the continuous Laguerre wavelet transform with help of Laguerre wavelet $\phi_{b,a}$ is defined by

$$\left(\mathrm{L}_\phi f\right)(b,a) = < \mathrm{f(t)},\ \phi_{\mathrm{b,a}}(t) >_\wedge$$

$$= \int_0^\infty \mathrm{f(t)}\ \overline{\phi_{\mathrm{b,a}}(t)}\ \mathrm{d}\wedge\mathrm{(t)} \tag{3.3}$$

$$= \int_0^\infty\int_0^\infty f(t)\ \overline{\phi(\mathrm{az})}\ \mathrm{d(b,t,z)}\ \mathrm{d}\wedge\mathrm{(z)}\ \mathrm{d}\wedge\mathrm{(t)} \tag{3.4}$$

Provided the integral is convergent. Since by (2.15) $\phi_{b,a} \in \mathrm{L}_{\mathrm{p},\wedge}$ whenever $\phi \in L_{p,\wedge}$.

The Laguerre wavelet transform can be expressed in the form of Laguerre transform as follows

$$\mathrm{L}\left[\left(\mathrm{L}_\phi f\right)(b,a)\right] = \hat{f}(n)\hat{\phi}(a,n). \tag{3.5}$$

Also, the Laguerre wavelet transform can be written as

$$\left(\mathrm{L}_\phi f\right)(b,a) = \left(f * \phi(a,.)\right)(b) . \tag{3.6}$$

IV. Plancherel and Parsevals Relation for Continuous Laguerre Wavelet Transform

This section describes significant properties of Continuous Laguerre Wavelet Transform, such as the Plancherel and inversion formula.

Theorem 4.1 (***Plancherel***) Let $f, g \in L_{2,\wedge}[0,\infty)$. Then we have

$$\left\langle \left(\mathrm{L}_\phi f\right)(b,a), \left(\mathrm{L}_\phi g\right)(b,a) \right\rangle_{L_{2,\wedge}[0,\infty)\times L_{2,\wedge}[0,\infty)} = c_{\phi_1,\phi_2} \left\langle f, g \right\rangle_{L_{2,\wedge}[0,\infty)} \tag{4.1}$$

Where,

$$c_{\phi_1,\phi_2} = \int_0^{\infty} \hat{\phi}_1(a,n)\hat{\phi}_2(a,n)\, d\wedge(a) \tag{4.2}$$

Proof. Let $f, g \in L_{2,\wedge}[0,\infty)$ then, we have

$$\int_0^{\infty}\int_0^{\infty} \left(\mathrm{L}_{\phi_1} f\right)(b,a)\overline{\left(\mathrm{L}_{\phi_2} g\right)(b,a)}\, d\wedge(a)\, d\wedge(b)$$

$$= \int_0^{\infty}\int_0^{\infty} \mathrm{L}^{-1}\left[\hat{f}(n)\hat{\phi}_1(a,n)\right](b)\,\overline{\mathrm{L}^{-1}\left[\hat{g}(n)\hat{\phi}_2(a,n)\right]}(b)\, d\wedge(a)\, d\wedge(b).$$

Now by using (2.10) we get

$$\int_0^{\infty}\int_0^{\infty} f(x)\phi_1(a,x)(b)\overline{g(x)\phi_2(a,x)(b)}\, d\wedge(b)\, d\wedge(a) = \sum_m \sigma(n)\hat{f}(n)\overline{\hat{g}(n)}\int_0^{\infty}\hat{\phi}_1(a,n)\hat{\phi}_2(a,n)\, d\wedge(a)$$

$$= c_{\phi_1,\phi_2}\sum_m \sigma(n)\hat{f}(n)\hat{g}(n).$$

Hence by using the Parseval formula for Laguerre transform, we get

$$\int_0^{\infty}\int_0^{\infty} \left(\mathrm{L}_{\phi_1} f\right)(b,a)\overline{\left(\mathrm{L}_{\phi_2} g\right)(b,a)}\, d\wedge(a)\, d\wedge(b) = c_{\phi_1,\phi_2}\int_0^{\infty} f(x) g(x)\, d\wedge(x)$$

$$= c_{\phi_1,\phi_2}\left\langle f, g\right\rangle_{L_{2,\wedge}[0,\infty)}.$$

Which completes the proof.

Theorem 5.2 (*Inversion Formula*) Let $f \in L_{2,\wedge}[0,\infty)$ and $\phi_{b,a}$ is Laguerre wavelet which defines Continuous Laguerre Wavelet Transform as (3.4). Then

$$f(x) = \frac{1}{c_{\phi}}\int_0^{\infty}\int_0^{\infty} \left(\mathrm{L}_{\phi} f\right)(b,a)\phi_{b,a}(t)\, d\wedge(a)\, d\wedge(b). \tag{4.3}$$

Where c_{ϕ} is given in (4.2).

Proof. Let $h(x) \in L_{2,\wedge}$ be any function, then by applying previous theorem, we have

$$c_{\psi}\left\langle f_1, h\right\rangle_{L_{2,\wedge}[0,\infty)} = \int_0^{\infty}\int_0^{\infty} \left(\mathrm{L}_{\phi} f_1\right)(b,a)\overline{\left(\mathrm{L}_{\phi} h\right)(b,a)}\, dadb$$

$$=\int_0^\infty\int_0^\infty\left(\mathrm{L}_\phi f_1\right)(b,a)\int_0^\infty h(t)\overline{\overline{\phi_{b,a}(t)}}d\wedge(t)d\wedge(a)d\wedge(b)$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\left(\mathrm{L}_\phi f_1\right)(b,a)\phi_{b,a}(t)\overline{h(t)}d\wedge(t)d\wedge(a)d\wedge(b)$$

$$=\int_0^\infty g(t)\overline{h(t)}d\wedge(t)$$

$$=\langle g,h\rangle_\wedge .$$

Where,

$$g(t)=\int_0^\infty\int_0^\infty\left(\mathrm{L}_\phi f_1\right)(b,a)\phi_{b,a}(t)d\wedge(a)d\wedge(b)$$

Thus, $c_\phi\langle f,h\rangle_\wedge=\langle g,h\rangle_\wedge$

$$f=\frac{1}{c_\phi}g=\frac{1}{c_\phi}\int_0^\infty\int_0^\infty\left(\mathrm{L}_\phi f_1\right)(b,a)\phi_{b,a}(t)d\wedge(a)d\wedge(b)$$

If $f=h$

$$\|f\|^2_{L_{2,\wedge}[0,\infty)}=\int_0^\infty\int_0^\infty\left|\left(\mathrm{L}_\phi f_1\right)(b,a)\right|^2d\wedge(a)d\wedge(b)$$

Moreover the Laguerre wavelet transform is isometry from $L_{2,\wedge}[0,\infty)$ to $L_{2,\wedge}[0,\infty)\times L_{2,\wedge}[0,\infty)$.

V. Concluding Remarks

This article introduces the concept of wavelet transform associated with generalized translation and convolutions structure. Moreover it discusses the concept of hyper geometric wavelets associated with special functions. The concepts presented in this article have a lot of potential for flourishing the wavelets on hepergroups. Therefore this article is very useful as it invites the researchers to work on it to explore more on harmonic analysis.

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