**Abstract**

In this paper, we introduce two notions of $T_1$ property in fuzzy topological spaces by using quasi-coincidence sense and we establish relationship among our and others such notions. We also show that all these notations satisfy good extension property. Also hereditary, productive and projective properties are satisfied by these notions. We observe that all these concepts are preserved under one-one, onto, fuzzy open and fuzzy continuous mappings. Finally, we discuss initial and final fuzzy topologies on our second notion.

**Keywords**: Fuzzy Topological Space, Quasi-coincidence, Fuzzy $T_1$ Topological Space, Good Extension.

**I. Introduction**

Chang [5] defined fuzzy topological spaces in 1968 by using fuzzy sets introduced by Zadeh [25] in 1965. Since then extensive work on fuzzy topological spaces has been carried out by many researchers like Gouguen [7], Wong [23], Lowen [13], Warren [22], Hutton [10] and others. Separation axioms are important parts in fuzzy topological spaces. Many works [6, 1, 8, 3, 19, 20] on separation axioms have been done by researchers. Among those axioms, fuzzy $T_1$ type is one and it has been already introduced in the literature. There are many articles on fuzzy $T_1$ topological space which are created by many authors like P. Wuyts and R. Lowen [24], D. M. Ali [1], Srivastava et al. [21] M. S. Hossain and D. M. Ali [9] and many others.

The purpose of this paper is to further contribute to the development of fuzzy topological spaces especially on fuzzy $T_1$ topological spaces. In the present paper, fuzzy $T_1$ topological space is defined by using quasi-coincidence sense and
relations among the given and other such notions are shown here. It is showed that the good extension property is satisfied on our notions. In the next section of this paper, it is also showed that the hereditary, order preserving, productive, and projective properties hold on the new concepts. Finally, we discuss initial and final fuzzy topologies on our second notion.

II. Basic notions and preliminary results

In this section, we recall some concepts occurring in the papers [1, 25] which will be needed in the sequel. In the present paper $X$ and $Y$ always denote non empty sets and $I=[0,1]$, $I_1=[0,1)$. The class of all fuzzy sets on a non empty set $X$ is denoted by $I^X$ and fuzzy sets on $X$ are denoted as $u, v, w$ etc. Crisp subsets of $X$ are denoted by $U, V, W$ etc. throughout this paper.

**Definition 2.1** [25] A function $u$ from $X$ into the unit interval $I$ is called a fuzzy set in $X$. For every $x \in X$, $u(x) \in I$ is called the grade of membership of $x$ in $u$.

**Definition 2.2** [16] A fuzzy set $u$ in $X$ is called a fuzzy singleton if and only if $u(x) = r, 0 < r \leq 1$ for a certain $x \in X$ and $u(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy singleton is denoted by $x_r$ and $x$ is its support. The class of all fuzzy singletons in $X$ will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$.

**Definition 2.4** [11] A fuzzy singleton $x_r$ is said to be quasi-coincidence with $u$, denoted by $x_r \sqsubseteq u$ if and only if $u(x) + r > 1$. If $x_r$ is not quasi-coincidence with $u$, we write $x_r \nsubseteq u$ and defined as $u(x) + r \leq 1$.

**Definition 2.5** [5] Let $f$ be a mapping from a set $X$ into a set $Y$ and $u$ be a fuzzy subset of $X$. Then $f$ and $u$ induce a fuzzy subset $v$ of $Y$ defined by $v(y) = \sup \{u(x)\}$ if $x \in f^{-1}[\{y\}] \neq \emptyset, x \in X = 0$ otherwise.

**Definition 2.6** [5] Let $f$ be a mapping from a set $X$ into a set $Y$ and $v$ be a fuzzy subset of $Y$. Then the inverse of $v$ written as $f^{-1}(v)$ is a fuzzy subset of $X$ defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

**Definition 2.7** [5] Let $I = [0, 1]$, $X$ be a non empty set and $I^X$ be the collection of all mappings from $X$ into $I$, i.e. the class of all fuzzy sets in $X$. A fuzzy topology on $X$ is defined as a family $t$ of members of $I^X$, satisfying the following conditions.

(i) $1, 0 \in t$,
(ii) If $u \in t$ for each $i \in \Lambda$ then $\bigcap_{i \in \Lambda} u_i \in t$, where $\Lambda$ is an index set.
(iii) If $u, v \in t$ then $u \cap v \in t$.

The pair $(X, t)$ is called a fuzzy topological space (in short fts) and members of $t$
are called $t$-open fuzzy sets. A fuzzy set $v$ is called a $t$-closed fuzzy set if $1 - v \in t$.

**Definition 2.8** [17] The function $f : (X, t) \to (Y, s)$ is called fuzzy continuous if and only if for every $v \in s, f^{-1}(v) \in t$, the function $f$ is called fuzzy homeomorphic if and only if $f$ is bijective and both $f$ and $f^{-1}$ are fuzzy continuous.

**Definition 2.9** [15] The function $f : (X, t) \to (Y, s)$ is called fuzzy open if and only if for every open fuzzy set $u$ in $(X, t)$, $f(u)$ is open fuzzy set in $(Y, s)$.

**Definition 2.12** [12] Let $(X_i, i \in \Lambda)$, be any class of sets and let $X$ denotes the Cartesian product of these sets, i.e., $X = \prod_{i \in \Lambda} X_i$. Note that $X$ consists of all points $p = (a_i, i \in \Lambda)$, where $a_i \in X_i$. Recall that, for each $j_0 \in \Lambda$, we define the projection $\pi_{j_0}$ from the product set $X$ to the coordinate space $X_{j_0}$, i.e., $\pi_{j_0} : X \to X_{j_0}$ by $\pi_{j_0}(a_i, i \in \Lambda) = a_{j_0}$.

**Definition 2.12** [23] Let $\{X_i, i \in \Lambda\}$ be a family of nonempty sets. Let $X = \prod_{i \in \Lambda} X_i$ be the usual product of $X_i$’s and let $\pi_i$ be the projection from $X$ into $X_i$. Further assume that each $X_i$ is a fuzzy topological space with fuzzy topology $t_i$. Now the fuzzy topology generated by $\{\pi_i^{-1}(b_i) : b_i \in t_i, i \in \Lambda\}$ as a sub basis, is called the product fuzzy topology on $X$. Clearly if $w$ is a basis element in the product, then there exist $i_1, i_2, i_3, \ldots, i_n \in \Lambda$, with $x = (x_i)_{i \in \Lambda} \in X$ such that $w(x) = \min\{b_i(x_i) : i = 1, 2, 3, \ldots, n\}$.

**Definition 2.14** [18] Let $f$ be a real valued function on a topological space. If $\{x : f(x) > \alpha\}$ is open for every real $\alpha$, then $f$ is called lower semi continuous function.

**Definition 2.15** [13] Let $X$ be a nonempty set and $T$ be a topology on $X$. Let $t = \omega(T)$ be the set of all lower semi continuous functions from $(X, T)$ to $I$ (with usual topology). Then $\omega(T) = \{u \in I^X : u - 1(\alpha, 1) \in T\}$ for each $\alpha \in I$. It can be shown that $\omega(T)$ is a fuzzy topology on $X$.

**Definition 2.15** [14] The initial fuzzy topology on a set $X$ for the family of fts $\{(X_i, t_i)\}_{i \in \Lambda}$ and the family of functions $\{f_i : X \to (X_i, t_i)\}_{i \in \Lambda}$ is the smallest fuzzy topology on $X$ making each $f_i$ fuzzy continuous. It is easily seen that it is generated by the family $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i \in \Lambda}$.

**Definition 2.16** [14] The final fuzzy topology on a set $X$ for the family of fts $\{(X_i, t_i)\}_{i \in \Lambda}$
and the family of functions \( \{f_i : (X_i, t_i) \to X \}_{i=k} \) is the finest fuzzy topology on X making each \( f_i \) fuzzy continuous.

**Theorem 2.1** [2] A bijective mapping from an fts \( (X, t) \) to an fts \( (Y, s) \) preserves the value of a fuzzy singleton (fuzzy point). Note: Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

### III. The Main Results

In this section, we discuss about our notions and findings. Some well-known properties are discussed here by using our concepts.

**Definition 3.1** A fuzzy topological space \( (X, t) \) is called

(a) \( FT_1(i) \) if and only if for any pair \( x, y \in X \) with \( x \neq y \), there exists \( u, v \in t \) such that \( x, q \cup y, q \cup u \) and \( y, q \cup x, q \cup v \).

(b) \( FT_1(ii) \) if and only if for any pair \( x, y \in X \) with \( x \neq y \), there exists \( u, v \in t \) such that \( x, q \cup u, y, \cap u = 0 \) and \( y, q \cup v, x, \cap v = 0 \).

(c) ([21]) \( FT_1(iii) \) if and only if for any pair \( x, y \in X \) with \( x \neq y \), there exists \( u, v \in t \) such that \( u(x) = 1, u(y) = 0 \) and \( v(y) = 1, v(x) = 0 \).

**Example 3.1** Let \( X = \{x, y\} \) and \( u \in t^X \) be given by \( u(x) = 1, u(y) = 0 \) and \( v(y) = 1, v(x) = 0 \). Let us consider the fuzzy topology \( t \) on \( X \) generated by \( \{0, u, v, 1\} \). Let \( x, y \) be fuzzy singletons in \( X \) with \( x \neq y \). Then \( u(x) + r \geq 1 \) and \( u(y) + s \leq 1 \) for \( r, s \in (0, 1] \). Therefore, \( x, q \cup y, q \cup u \). Similarly, \( y, q \cup x, q \cup v \). This shows that \( (X, t) \) is \( FT_1(i) \). Also as \( u(y) = 0 \) and \( v(x) = 0 \), \( y, \cap u = 0 \) and \( x, \cap v = 0 \). Thus \( (X, t) \) is \( FT_1(ii) \).

**Lemma 3.1** For a fuzzy topological space \( (X, t) \) the following implications are true:

\[ FT_1(ii) \Rightarrow FT_1(i), \quad FT_1(iii) \Rightarrow FT_1(i), \quad FT_1(iii) \Rightarrow FT_1(i). \]

But in general the converse is not true.

**Proof:** \( FT_1(ii) \Rightarrow FT_1(i) \): Let \( (X, t) \) be a fuzzy topological space and \( (X, t) \) is \( FT_1(ii) \). We have to prove that \( (X, t) \) is \( FT_1(i) \). Let \( x, y \) be fuzzy singletons in \( X \) with \( x \neq y \). Since \( (X, t) \) is \( FT_1(ii) \) fuzzy topological space, there exists \( u, v \in t \) such that \( x, q \cup u, y, \cap u = 0 \) and \( y, q \cup v, x, \cap v = 0 \). To prove \( (X, t) \) is \( FT_1(i) \), it is only needed to prove that \( y, q \cup u \) and \( x, q \cup v \).

Now, \( y, \cup u = 0 \Rightarrow u(y) = 0 \Rightarrow u(y) + s \leq 1 \Rightarrow y, q \cup u \). Similarly, we can prove that \( x, q \cup v \). Hence, \( (X, t) \) is \( FT_1(i) \). To show \( FT_1(i) \neq FT_1(ii) \), we give a counter example.

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Counter example: Let $X = \{x, y\}$ and $u, v \in \mathcal{F}(X)$ be given by $u(x) = 1, u(y) = 0.1, v(y) = 1, v(x) = 0.1$. Let us consider the fuzzy topology $t$ on $X$ generated by $\{0, u, v, 1\}$. For $0 < r \leq 1, 0 < s < 0.9$, $u(x) + r > 1 \Rightarrow x_r u$ and $u(y) + s \leq 1 \Rightarrow y_s \bar{u}$. Hence it is clear that $(X, t)$ is $FT_\mathcal{D}(i)$. But $u(y) \neq 0$ and $v(x) \neq 0 \Rightarrow y_s \cap u \neq 0$. Hence it is clear that $(X, t)$ is not $FT_\mathcal{D}(ii)$.

**Proof:** $FT_\mathcal{D}(iii) \Rightarrow FT_\mathcal{D}(i)$: Let $(X, t)$ be a fuzzy topological space and $(X, t)$ is $FT_\mathcal{D}(iii)$. We have to prove that $(X, t)$ is $FT_\mathcal{D}(i)$. Let $x_r, y_s$ be fuzzy singletons in $X$ with $x \neq y$.

Since $(X, t)$ is $FT_\mathcal{D}(iii)$ fuzzy topological space, there exists $u, v \in t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. To prove $(X, t)$ is $FT_\mathcal{D}(i)$, it is needed to prove that $x_r u y_s \bar{u}$ and $y_s \bar{u} x_r \bar{u}$. Now, $u(x) = 1 \Rightarrow u(x) + r > 1 \text{ for any } r \in (0,1] \Rightarrow x_r u$ and $u(y) = 0 \Rightarrow u(y) + s \leq 1 \text{ for any } s \in (0,1] \Rightarrow y_s \bar{u}$. Similarly, we can prove that $y_s \bar{u} x_r \bar{u}$. Hence, $(X, t)$ is $FT_\mathcal{D}(i)$. To show $FT_\mathcal{D}(i) \neq FT_\mathcal{D}(iii)$, we give a counter example.

**Counter example:** Let $X = \{x, y\}$ and $u, v \in \mathcal{F}(X)$ be given by $u(x) = 1 - \epsilon u(y) = 0, v(y) = 1 - \epsilon v(x) = 0$, where $\epsilon = \frac{r}{2}$ for $r \in (0,1]$. Let us consider the fuzzy topology $t$ on $X$ generated by $\{0, u, v, 1\}$. Then $u(x) = 1 - \epsilon \Rightarrow u(x) + \frac{r}{2} = 1 \Rightarrow u(x) + r > 1 \Rightarrow x_r u$ and $u(y) + s \leq 1 \Rightarrow y_s \bar{u}$. Similarly, we can prove that $y_s \bar{u} x_r \bar{u}$. Hence, $(X, t)$ is $FT_\mathcal{D}(i)$. But $u(x) \neq 1$ and $v(y) \neq 1$. Hence, $(X, t)$ is not $FT_\mathcal{D}(iii)$.

**Proof:** $FT_\mathcal{D}(iii) \Rightarrow FT_\mathcal{D}(ii)$: Let $(X, t)$ be a fuzzy topological space and $(X, t)$ is $FT_\mathcal{D}(iii)$. We have to prove that $(X, t)$ is $FT_\mathcal{D}(ii)$. Let $x_r, y_s$ be fuzzy singletons in $X$ with $x \neq y$.

Since $(X, t)$ is $FT_\mathcal{D}(iii)$ fuzzy topological space, there exists $u, v \in t$ such that $u(x) = 1, u(y) = 0$ and $v(y) = 1, v(x) = 0$. To prove $(X, t)$ is $FT_\mathcal{D}(ii)$, it is needed to prove that $x_r u y_s \cap u = 0$ and $y_s \bar{u} x_r \cap v = 0$.

Now, $u(x) = 1 \Rightarrow u(x) + r > 1 \text{ for any } r \in (0,1] \Rightarrow x_r u$ and $u(y) = 0 \Rightarrow y_s \cap u = 0 \text{ for any } s \in (0,1]$. Similarly, we can prove that $y_s \bar{u} x_r \cap v = 0$. Hence, $(X, t)$ is $FT_\mathcal{D}(ii)$. To show $FT_\mathcal{D}(ii) \neq FT_\mathcal{D}(iii)$, we give a counter example.
Counter example: Let \( X = \{x, y\} \) and \( u, v \in F^X \) be given by
\[
u(x) = 1 - \varepsilon \quad \nu(y) = 0,
\]
where \( \varepsilon = \frac{r}{2} \) for \( r \in (0, 1) \). Let us consider the fuzzy topology \( t \) on \( X \) generated by \( \{0, u, v, 1\} \). Then
\[
u(x) = 1 - \frac{r}{2} \Rightarrow u(x) + \frac{r}{2} = 1 \Rightarrow u(x) + r > 1 \Rightarrow x, q u \quad \text{and}
\]
\[
u(y) = 0 \Rightarrow \gamma y \cap u = 0.
\]
Similarly, we can prove that \( y_s q v, x_r \cap v = 0 \). Hence, \( (X, t) \) is \( FT_1(i) \) But \( u(x) \neq 1 \) and \( v(y) \neq 1 \). Hence, \( (X, t) \) is not \( FT_1(iii) \). These complete the proof of the implications.

Now we shall show that our notions satisfy the good extension property.

**Theorem 3.1** Let \((X, T)\) be a fuzzy topological space. Consider the following statements:

1. \((X, T)\) be a \( T_1 \) Topological Space.
2. \((X, \omega(T))\) be an \( FT_1(i) \) space.
3. \((X, \omega(T))\) be an \( FT_1(ii) \) space.

The implications: \((1) \iff (2) \quad (1) \iff (3)\) are true.

**Proof of (1) \iff (2):** Let \((X, T)\) be a topological space and \((X, T)\) is \( T_1 \). We have to prove that \((X, \omega(T))\) is \( FT_1(i) \). Let \( x_r, y_s \) be fuzzy points in \( X \) with \( x \neq y \). Since \((X, T)\) is \( T_1 \) topological space, we have, there exists \( U, V \in T \) such that \( x \in U \), \( y \notin U \) or \( y \in V \), \( x \notin V \). From the definition of lower semi continuous we have 
\[
1_u, 1_v \in \omega(T) \quad \text{and} \quad 1_u(x) = 1, \quad 1_v(y) = 0 \quad \text{and} \quad 1_v(y) + s \leq 1 \Rightarrow y_s q 1_u \quad \text{and} \quad 1_u(y) + r > 1 \Rightarrow x_r q 1_v
\]
It follows that there exists \( 1_u, 1_v \in \omega(T) \) such that \( x_r q 1_u, y_s q 1_v \). Hence \((X, \omega(T))\) is \( FT_1(i) \). Thus \((1) \Rightarrow (2) \) holds.

Conversely, let \((X, \omega(T))\) be a fuzzy topological space and \((X, \omega(T))\) is \( FT_1(i) \). We have to prove that \((X, T)\) is \( T_1 \). Let \( x_r, y_r \) be points in \( X \) with \( x \neq y \). Since \((X, \omega(T))\) is \( FT_1(i) \) topological space we have, for any fuzzy points \( x_r, y_r \) in \( X \), there exists \( u \in T \) such that \( x_r q u y_r q u \) and there exists \( v \in T \) such that \( y_r q v \).

Now, \( x_r q u \Rightarrow u(x) + r > 1 \Rightarrow u(x) > 1 - r = a \Rightarrow x \in u^{-1}(a, 1] \)
and \( y_r q v \Rightarrow u(y) + r \leq 1 \Rightarrow u(y) \leq 1 - r = a \Rightarrow u(y) \leq a \Rightarrow y \notin u^{-1}(a, 1] \)

Also \( u^{-1}(a, 1] \in T \). It follows that \( \exists u^{-1}(a, 1] \in T \) such that \( x \in u^{-1}(a, 1], y \notin u^{-1}(a, 1] \).

Thus \((2) \Rightarrow (1) \) holds. Similarly, we can prove that \((1) \iff (3)\)
Now we shall show that our notions satisfy the hereditary property.

**Theorem 3.2** Let \((X, t)\) be a fuzzy topological space, then

\[ A \subseteq X, t_A = \{ u/A : u \in t \} \]

(a) \((X, t)\) is \(FT_1(i) \Rightarrow (A, t_A)\) is \(FT_1(i)\) and

(b) \((X, t)\) is \(FT_1(ii) \Rightarrow (A, t_A)\) is \(FT_1(ii)\).

**Proof of (a):** Let \((X, t)\) be a fuzzy topological space and \((X, t)\) is \(FT_1(i)\). Let \(x_r, y_s\) be fuzzy points in \(A\) with \(x \neq y\). Since, \(A \subseteq X\) these fuzzy points are also fuzzy points in \(X\). Also since \((X, t)\) is \(FT_1(i)\) fuzzy topological space, we have, there exists \(u, v \in t\) such that \(x_rqu, y_s\bar{q}u\) and \(y_sqv, x_r\bar{q}v\). For \(A \subseteq X\), we have \(u/A \in t_A\).

Now, \(x_rqu \Rightarrow u(x) + r > 1, x \in X \Rightarrow u/A(x) + r > 1, x \in A \subseteq X \Rightarrow x_rqu/A\)

And

\[ y_s\bar{q}u \Rightarrow u(y) + s \leq 1, y \in X \Rightarrow u/A(y) + s \leq 1, y \in A \subseteq X \Rightarrow y_s\bar{q}u/A\]

Similarly, we can prove that \(y_sqv/A\) and \(x_r\bar{q}v/A\). Hence, \((A, t_A)\) is \(FT_0(i)\).

Proof of (b) is similar to proof of (a).

Now, we shall show that our notions satisfy the productive and projective properties.

**Theorem 3.3** Let \((X_i, t_i), i \in A\) be fuzzy topological spaces and \(X = \prod_{i \in A} \in X_i\)

and \(t\) be the product topology on \(X\), then

(a) for all \(i \in A\), \((X_i, t_i)\) is \(FT_1(i)\) if and only if \((X, t)\) is \(FT_1(i)\) and

(b) for all \(i \in A\), \((X_i, t_i)\) is \(FT_1(ii)\) if and only if \((X, t)\) is \(FT_1(ii)\).

**Proof of (b):** Let for all \(i \in A\), \((X_i, t_i)\) is \(FT_1(ii)\) space. We have to prove that \((X, t)\) is \(FT_1(ii)\). Let \(x_r, y_s\) be fuzzy points in \(X\) with \(x \neq y\). Then \((x_i)_r(y_i)_s\) are fuzzy points with \(x_i \neq y_i\) for some \(i \in A\). Since \((X_i, t_i)\) is \(FT_1(ii)\), there exists \(u_i, v_i \in t_i\) such that \((x_i)_rqu_i, (y_i)_s \cap u_i = 0\) and \((y_i)_sqv_i, (x_i)_r \cap v_i = 0\). Let \((x_i)_rqu_i, (y_i)_s \cap u_i = 0\). But we have \(\pi_i(x) = x_i\) and \(\pi_i(y) = y_i\)

Now, \((x_i)_rqu_i \Rightarrow u_i(x_i) + r > 1, x \in X \Rightarrow u_i(\pi_i(x)) + r > 1\)

\[ (u_i \circ \pi_i)(x) + r > 1 \Rightarrow x_rqu(x_i \circ \pi_i) \]

and \(y_i) q u_i = 0 \Rightarrow u_i(y_i) = 0, y \in X \Rightarrow u_i(\pi_i(y)) = 0, y \in X \Rightarrow (u_i \circ \pi_i)(y) = 0 \Rightarrow y_s \cap (u_i \circ \pi_i) = 0\) It follows that there exists
such that \( x_r q (u_i \circ \pi_i), y_s \cap (u_i \circ \pi_i) = 0. \)

Hence it is clear that \((X, t)\) is \(FT_1(\text{ii}).\)

Conversely, Let \((X, t)\) be a fuzzy topological space and \((X, t)\) is \(FT_1(\text{ii}).\) We have to prove that \((X_i, t_i), \ i \in A\) is \(FT_1(\text{ii}).\) Here let us consider, \(a_i\) be a fixed element in \(X_i.\)

Let

\[ A_i = \{ x \in X = \prod_{i \in A} X_i; \ x_j = \text{a}_j \text{ for some } i \neq j \}. \]

Then \(A_i\) is a subset of \(X,\) and hence \((A_i, t_{A_i})\) is a subspace of \((X, t).\) Since \((X, t)\) is \(FT_1(\text{ii}),\)

so \((A_i, t_{A_i})\) is \(FT_1(\text{ii}).\) Now we have \(A_i\) is homeomorphic image of \(X_i.\) Hence it is clear that for all \(i \in A\), \((X_i, t_i)\) is \(FT_1(\text{ii}).\) space. Thus (b) holds. Proof of (a) is similar to proof of (b).

Now, we shall show that our notions satisfy the order preserving property.

**Theorem 3.4** Let \((X, t)\) and \((Y, s)\) be two fuzzy topological spaces and \(f: X \rightarrow Y\) be a one-one, onto and fuzzy open map then

\((a)\) \((X, t)\) is \(FT_1(\text{i}) \Rightarrow (Y, s)\) is \(FT_1(\text{i})\) and \((b)(X, t)\) is \(FT_1(\text{ii}) \Rightarrow (Y, s)\) is \(FT_1(\text{ii}).\)

**Proof of (a):** Let \((X, t)\) be a fuzzy topological space and \((X, t)\) is \(FT_1(\text{i}).\) We have to prove that \((Y, s)\) is \(FT_1(\text{i}).\) Let \(x^\prime, y^\prime\) be fuzzy points in \(Y\) with \(x \neq y\). Since \(f\) is onto then there exist \(x, y \in X\) with \(f(x) = x^\prime, f(y) = y^\prime\) and \(x_r, y_s\) are fuzzy points in \(X\) with \(x \neq y\) as \(f\) is one-one. Again since \((X, t)\) is \(FT_1(\text{i})\) space, there exists \(u, v \in t\) such that \(x_r u, y_s \bar{u}\) and \(y_s q v, x_r \bar{q} v.\)

Let \(x_r u, y_s \bar{u}.\) Now, \(x_r u \Rightarrow u(x) + r > 1\) and \(y_s \bar{u}\)

\(= f(u(x), y^\prime) = u(y) + s \leq 1\)

\(\Rightarrow f(u(x)^\prime) = u(x),\) for some \(x\) and \(f(u(y)^\prime) = \sup u(y) = y^\prime = f(u(y)^\prime) = u(y),\) for some \(y.\) Also we have, \(f\) is a fuzzy open mapping. Then

\(f(u) \in s\) as \(u \in t.\)

Again, \(u(x) + r > 1 \Rightarrow f(u)(x^\prime) + r > 1 \Rightarrow x_r q f(u)\)

And

\(u(y) + s \leq 1 \Rightarrow f(u)(y^\prime) + s \leq 1 \Rightarrow y_s \bar{q} f(u).\) It follows that there exists \(f(u), f(v) \in s\) such that \(x_r q f(u), y_s \bar{q} f(u).\) Hence, \((Y, s)\) is \(FT_1(\text{i})\) space. Proof of (b) is similar to proof of (a).
**Theorem 3.5** Let \((X, t)\) and \((Y, s)\) be two fuzzy topological spaces and \(f : X \to Y\) be a one-one, onto and fuzzy continuous mapping then, (a) \((Y, s)\) is \(FT_1(i)\) \(\Rightarrow\) \((X, t)\) is \(FT_1(i)\) and (b) \((Y, s)\) is \(FT_1(ii)\) \(\Rightarrow\) \((X, t)\) is \(FT_1(ii)\).

**Proof of (b):** Let \((Y, s)\) be a fuzzy topological space and \((Y, s)\) is \(FT_1(ii)\). We have to prove that \((X, t)\) is \(FT_1(ii)\). Let \(x_r, y_s\) be fuzzy points in \(X\) with \(x \neq y\). Then \((f(x))_r, (f(y))_s\) are fuzzy points in \(Y\) with \(f(x) \neq f(y)\) as \(f\) is one-one. Again since \((Y, s)\) is \(FT_1(ii)\) space, there exists \(u, v \in s\) such that 
\[(f(x))_r q u, (f(y))_s \cap u = 0\]
and 
\[(f(y))_s q v, (f(x))_r \cap v = 0.\]
Let 
\[(f(x))_r q u, (f(y))_s \cap u = 0\]
Now, 
\[(f(x))_r q u \Rightarrow u(f(x)) + r > 1 \Rightarrow f^{-1}(u(x)) + r > 1 \Rightarrow f^{-1}((u))(x) + r > 1 \Rightarrow x_r q f^{-1}(u)\text{ and.} \]
Now, since, \(f\) is fuzzy continuous mapping and \(u, v \in s\) then \(f^{-1}(u), f^{-1}(v) \in t\). It follows that there exists \(f^{-1}(u), f^{-1}(v) \in t\) such that 
\(x_r q f^{-1}(u), y_s \cap f^{-1}(u) = 0\) and \(y_s q f^{-1}(v), x_r \cap f^{-1}(v) = 0\). Hence it is clear that \((X, t)\) is \(FT_0(ii)\) space. Proof of (a) is similar to proof of (b).

As our next work, here we introduce two theorems on the second notion of us. The idea of these theorems are taken from M. R. Amin and M. S. Hossain [4].

**Theorem 3.6** If \(\{X_i, t_i\}_{i \in \Lambda}\) is a family of \(FT_1(ii)\) fits and \(\{f_i : X \to (X_i, t_i)\}_{i \in \Lambda}\), a family of one-one and fuzzy continuous functions, then the initial fuzzy topology on \(X\) for the family \(\{f_i\}_{i \in \Lambda}\) is \(FT_1(ii)\).

**Proof:** Let \(t\) be the initial fuzzy topology on \(X\) for the family \(\{f_i\}_{i \in \Lambda}\). Let \(x_r, y_s\) be fuzzy points in \(X\) with \(x \neq y\). Then \(f_i(x), f_i(y) \in X_i\) and \(f_i(x) \neq f_i(y)\) as \(f_i\) is one-one. Since \((X_i, t_i)\) is \(FT_1(ii)\) then for every two distinct fuzzy points \((f_i(x))_r, (f_i(y))_s\) in \(X_s\), there exist fuzzy sets \(u_i \in t_i\) such that \((f_i(x))_r q u_i, (f_i(y))_s \cap u_i = 0\) and \((f_i(y))_s q v_i, (f_i(x))_r \cap v_i = 0\). Now, \((f_i(x))_r q u_i\) and \((f_i(y))_s \cap u_i = 0\).

That is \(u_i(f_i(x)) + r > 1\) and \(u_i(f_i(y)) = 0\). That is \(f_i^{-1}(u_i)(x) + r > 1\) and \(f_i^{-1}(u_i)(y) = 0\). This is true for every \(i \in \Lambda\). So inf \(f_i^{-1}(u_i)(x) + r > 1\) and inf \(f_i^{-1}(u_i)(y) = 0\). Let \(u = \text{inf} f_i^{-1}(u_i)\). Then \(u \in t\) as \(f_i\) is fuzzy continuous. So \(u(x) + r > 1\) and \(u(y) = 0\). Hence \(x_r, q u\) and \(y_s \cap u = 0\). Similarly, we can prove that \(y_s q u\) and \(x_r \cap v = 0\). Therefore, \((X, t)\) is \(FT_1(ii)\).
Theorem 3.7 If \( \{(X_i, t_i)\}_{i \in \Lambda} \) is a family of \( FT_1 \) fts and \( \{f_i: (X_i, t_i) \rightarrow X\}_{i \in \Lambda} \) a family of fuzzy open and bijective function, then the final fuzzy topology on \( X \) for the family \( \{f_i\}_{i \in \Lambda} \) is \( FT_1 \).

Proof: Let \( t \) be the final fuzzy topology on \( X \) for the family \( \{f_i\}_{i \in \Lambda} \). Let \( x, y \) be fuzzy points in \( X \) with \( x \neq y \). Then \( f_i^{-1}(x), f_i^{-1}(y) \in X_i \) and \( f_i^{-1}(x) \neq f_i^{-1}(y) \) as \( f_i \) is bijective. Since \( (X_i, t_i) \) is \( FT_1 \) then for every two distinct fuzzy points \( ( f_i^{-1}(x) )_r, ( f_i^{-1}(y) )_s \) in \( X \), there exist fuzzy sets \( u_i, v_i \in t_i \) such that \( ( f_i^{-1}(x) )_r \cap u_i = 0 \) and \( ( f_i^{-1}(y) )_s \cap v_i = 0 \). Now \( ( f_i^{-1}(x) )_r \cap u_i \cap ( f_i^{-1}(y) )_s \cap v_i = 0 \) and \( u_i \left( f_i^{-1}(x) \right) + r > 1 \) and \( u_i \left( f_i^{-1}(y) \right) = 0 \). That is \( f_i(u_i) = 0 \). This is true for every \( i \in \Lambda \). So \( \inf f_i(u_i) = 0 \). Let \( u = \inf f_i(u_i) \). Then \( u \in t \) as \( f_i \) is fuzzy open. So, \( u(x) + r > 1 \) and \( u(y) = 0 \). Hence \( x, u \) and \( y, u \) are fuzzy open. Similarly, we can prove that \( y_u \) and \( x, u \) and \( y, u \) are fuzzy open. Therefore, \( (X, t) \) is \( FT_1 \).

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