Abstract

In this article, we investigate the traveling wave solutions to the Klein-Gordon equation in (2+1)-dimension with special types of nonlinearity. The types include quadratic, cubic and polynomial nonlinearity. The Klein-Gordon equation assumes significant role in numerous types of scientific investigation such as in quantum field theory, nonlinear optics, nuclear physics, magnetic field etc. To investigate the aimed traveling wave solutions, we execute the \( \left( \frac{G'}{G} \right) \)-expansion method. The attained solutions are in the form of hyperbolic, trigonometric and rational functions. The results acknowledged that the applied method is very efficient and suitable for discovering differential equations with various types of nonlinearity considered in optics and quantum field theory. The solutions of the Klein-Gordon equation with quadratic, cubic, and polynomials nonlinearity play a significant role in many scientific measures notably optics and quantum field theory.

Keywords: Klein-Gordon equation; nonlinearity; traveling wave solutions

I. Introduction

In 1926, physicists Oskar Klein and Walter Gordon suggested an equation which demonstrates about relativistic electrons and the equation is named after them as the Klein-Gordon equation. The Klein-Gordon equation is also measured as one of the version of Schrodinger equation which is a relativistic wave equation. The Klein-Gordon equation plays important role in many types scientific studies such as nonlinear optics, quantum field theory, nuclear physics, magnetic field, general relativity, solid state physics, and physical properties of matter. In nonlinear optics, the Klein Gordon equation considered as a vital part for Bose-Einstein condensates confined in strong optical lattices, which is shaped by the interference patterns of laser beam [VII, IX, XXII]. The solutions of Klein-Gordon equation contains
quantum scalar field where the particles are spin-less both in classical and quantum field theory. The Klein-Gordon equation is the most commonly investigated equation for explaining the particle dynamics in quantum field theory in comparison to other existing relativistic wave equations [VI, VII, IX, XXII, XVIII].

The Klein-Gordon equation is measured as nonlinear evolution equation (NLEE). The study of the exact traveling wave solutions of NLEEs plays a significant role to unveil the unexplored solutions of nonlinear phenomena. As a result, many new methods have been successfully developed to accomplish the solutions of these NLEEs, for instance, the sine-cosine method [XXI], the tanh function method [I], the solitary wave ansatz method [XII], the Lie symmetry method [VIII], the Adomian decomposition method [III, XX], the inverse scattering transform method [II], the modified simple equation method [V, XII], the residual power series method [XIII], the Lagrange characteristic method [XI], the Exp-function method [IV, X, XV], the F-expansion method [XIV] and so on. In this article, the \((G'/G)\)-expansion method will be executed to examine the new general closed form wave solutions to the Klein-Gordon equation in (2+1)-dimensions, where two-spatial dimensions, specifically in \(x\) and \(y\) coordinates and one-time dimension are exist. The solutions of the Klein-Gordon equation will address five types of nonlinearity.

II. Methodology

In this section, we interpret the generalized \((G'/G)\)-expansion method to investigate the traveling wave solutions of nonlinear evolution equations. Let us consider the general nonlinear evolution equation in the following form:

\[
P(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{tt}, u_{tx}, \ldots) = 0,
\]

where, \(u = u(x, y, t)\) is an unknown function, \(P\) is a polynomial in \(u(x, y, t)\) and its derivative wherein the maximum order of derivatives and nonlinear terms are associated and the subscripts stand for the partial derivatives. In order to attain the solitary wave solutions of the Klein-Gordon equations, we have to execute the following steps:

Step 1: Suppose the traveling wave variable

\[
u(x, y, t) = u(\xi), \quad \xi = x + y - \omega t
\]

where, \(\omega\) is the speed of the traveling wave. The wave variable (2.2) permits us to change the Eq. (2.1) into an ordinary differential equation (ODE) for \(u = u(\xi)\):

\[
Q(u, u', u'', u''', \ldots) = 0,
\]

(2.3)
where, $Q$ is a function of $u(\xi)$ and its derivatives and the superscripts indicate the ordinary derivatives with respect to $\xi$.

**Step 2:** We suppose that the Eq. (2.3) has the solution in the form

$$u(\xi) = \sum_{r=0}^{m} a_r \left[ \frac{G'(\xi)}{G(\xi)} \right]^r,$$

where $G = G(\xi)$ satisfies the following ordinary differential equation in which $a_k \ (r = 0, 1, 2, 3, \ldots)$, such that $a_n \neq 0$ are arbitrary constants to be determined.

$$G'' + \lambda G' + \mu G = 0,$$  

(2.5)

in which where $\lambda$ and $\mu$ are arbitrary constants.

**Step 3:** We find the value of positive integer $m$ in Eq. (2.5) by considering the homogenous balanced between the highest order derivatives and the nonlinear terms in Eq. (2.5).

**Step 4:** In the case, if the value does not converted to an integer, we consider $u = v^m$ which transforms the Eq. (2.6) to a new equation in which $m$ becomes an integer.

**Step 5:** We place Eq. (2.7) along with Eq. (2.5) into Eq. (2.8) and collect all terms of the same power of $(G'/G)$ together, the left-hand side of Eq. (2.9) is then transformed to another polynomial in $(G'/G)$. Equating each coefficient of the resulted polynomial to zero, gives a set of algebraic equations for $a_n, \mu, \lambda$ and $\omega$. We solve the set of equations for the value of $a_n, \mu, \lambda$ and $\omega$.

**Step 6:** From Eq. (2.6) we can get the following general solutions

When

$$\lambda^2 - 4\mu > 0,$$

(2.6)

$$G = e^{-\frac{\lambda}{2} \xi} \left( c_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right);$$

When

$$\lambda^2 - 4\mu < 0,$$

(2.7)

$$G = e^{-\frac{\lambda}{2} \xi} \left( c_1 \cos \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \sin \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right);$$

When

$$\lambda^2 - 4\mu = 0,$$

(2.8)

$$G = (c_1 + c_2 \xi)e^{-\frac{\lambda}{2} \xi};$$
wherein \( c_1 \) and \( c_2 \) are arbitrary constants. Finally, using Eqs. (2.6)- (2.8) and Eq. (2.10) together with the values of arbitrary constants \( a_n \) and traveling wave speed \( \omega \) (achieved in step 5), we obtain the traveling wave solutions of Eq. (2.11).

III. Formulation of the Solutions

In this section, we implement the generalized \((G'/G)\)-expansion method discussed in Section 0 to investigate the Klein-Gordon equation with five types of nonlinearity namely quadratic, cubic and three polynomial nonlinear terms. The method induced new general exact closed-form solutions to the Klein-Gordon equation where two-spatial dimensions and one-time dimension prevail. The Klein-Gordon equation in (2+1)-dimension can be written as [VII, XVII, XIX, XXIII]:

\[
\ddot{u} - k^2 (u_{xx} + u_{yy}) + F(u) = 0 \tag{3.1}
\]

In the above Eq. (3.1), \( k \) is a real number where the dependent variable \( u(x,y,t) \) represents the quantized field explaining the spin-less particle. In Eq. (3.1) function \( F(u) \) is continuous where \( F(u) \) can be written in terms of the prospective function \( P(u) \) as [VII, XVI]:

\[
F(u) = -\frac{\partial P}{\partial u} \tag{3.2}
\]

The solutions to Eq. (3.1) for different forms of the nonlinear functions, \( F \) is known as the soliton. Eq. (3.1) provides both non-topological and topological solitons. The following five forms of the functions are considered that will give the different forms of soliton solutions which are as follows:

\[
F(u) = au - bu^2 \tag{3.3}
\]
\[
F(u) = au - bu^3 \tag{3.4}
\]
\[
F(u) = au - bu^n \tag{3.5}
\]
\[
F(u) = au - bu^n + cu^{2n-1} \tag{3.6}
\]
\[
F(u) = au - bu^{1-n} + cu^{n+1} \tag{3.7}
\]

These five cases will be leveled as Types I-V, where \( a, b \) and \( c \) are real valued number in all of these five types.

Type I

In this case, by means of Eq. (3.1) and (0.3), the Klein-Gordon equation in (2+1)-dimension can be written as
\[ u_{tt} - k^2 (u_{xx} + u_{yy}) + au - bu^2 = 0 \]  
(3.8)

The above equation also recognized quadratic nonlinear Klein-Gordon equation in (2+1)-dimension \[ [VII, XVI] \]. For investigating Eq. (3.8), we consider the subsequent transformation:

\[ u(x, y, t) = u(\xi) \]  
(3.9)

where

\[ \xi = x + y - \omega t \]  
(3.10)

Here \( \omega \) is the velocity of the soliton. Using Eq. (3.9) and (3.10), Eq. (3.8) converts into an ODE:

\[ (\omega^2 - 2k^2)u'' + au - bu^2 = 0 \]  
(3.11)

Balancing the highest derivatives of linear term \( u'' \) and nonlinear term of the highest order \( u^2 \), yields \( m = 2 \). By using \( (G'/G) \)-expansion method, we search out the solution in the following form

\[ u = a_0 + a_1 (G'/G) + a_2 (G'/G)^2, \]  
(3.12)

where \( a_0, a_1 \) and \( a_2 \) are constants to be determined.

We substitute Eq. (3.12) together with Eqs. (2.5)-(2.8) into Eq. (3.11) yields an over-determined set of algebraic equations. Solving these algebraic equations with the help of symbolic computation software, we obtain the following two sets of values for each constant appears in Case 1 and Case 2.

**Case 1:**

\[ \omega = \pm \frac{\sqrt{(4\mu - \lambda^2)(a + 8\mu k^2 - 21k^2)}}{4\mu - \lambda^2}, \quad a_0 = \frac{6a\mu}{(4\mu - \lambda^2)b}, \quad a_1 = \frac{6a\lambda}{(4\mu - \lambda^2)b}, \quad a_2 = \frac{6a}{(4\mu - \lambda^2)b} \]  
(3.13)

When \( \lambda^2 - 4\mu > 0 \), then using the values from Eq. (3.13) into Eq. (3.12), we obtain the following hyperbolic function solutions:
\[ u = \frac{6a\mu}{(4\mu - \lambda^2)b} + \frac{6a\lambda}{(4\mu - \lambda^2)b} \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{c_1 \sinh \frac{\lambda^2 - 4\mu}{2} + c_2 \cosh \frac{\lambda^2 - 4\mu}{2}}{c_1 \cosh \frac{\lambda^2 - 4\mu}{2} + c_2 \sinh \frac{\lambda^2 - 4\mu}{2}} \right) \right) + \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{c_1 \sinh \frac{\lambda^2 - 4\mu}{2} + c_2 \cosh \frac{\lambda^2 - 4\mu}{2}}{c_1 \cosh \frac{\lambda^2 - 4\mu}{2} + c_2 \sinh \frac{\lambda^2 - 4\mu}{2}} \right) \right)^2 \] (3.14)

Since \( c_1 \) and \( c_2 \) are arbitrary constants, one may arbitrarily choose their values. Therefore, if we choose \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) respectively and simplifying, we obtain

\[ u = -\frac{3a}{2b} \csc \left( \frac{\lambda^2 - 4\mu}{2} \right) \] \hspace{1cm} (3.15)

\[ u = \frac{3a}{2b} \sec \left( \frac{\lambda^2 - 4\mu}{2} \right) \] \hspace{1cm} (3.16)

Now we consider the condition \( \lambda^2 - 4\mu < 0 \), and applying the similar procedure the outcomes from Eq. (0.13) into Eq. (0.12), we obtain the following trigonometric solutions

\[ u = \frac{6a\mu}{(4\mu - \lambda^2)b} + \frac{6a\lambda}{(4\mu - \lambda^2)b} \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{c_1 \sin \frac{4\mu - \lambda^2}{2} + c_2 \cos \frac{4\mu - \lambda^2}{2}}{c_1 \cos \frac{4\mu - \lambda^2}{2} + c_2 \sin \frac{4\mu - \lambda^2}{2}} \right) \right) + \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{c_1 \sin \frac{4\mu - \lambda^2}{2} + c_2 \cos \frac{4\mu - \lambda^2}{2}}{c_1 \cos \frac{4\mu - \lambda^2}{2} + c_2 \sin \frac{4\mu - \lambda^2}{2}} \right) \right)^2 \] \hspace{1cm} (3.17)

As \( c_1 \) and \( c_2 \) are integral constants, one may freely set their values. Therefore, if we set \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) correspondingly, we obtain
\[ u = \frac{3a}{2b} \left( 1 - \cot^2 \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right) \]  
(3.18)

\[ u = \frac{3a}{2b} \left( 1 - \tan^2 \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right) \]  
(3.19)

It is noteworthy to mention that the solutions (3.15), (3.16), (3.18) and (3.19) are free of the arbitrary constant which promises to gain some known solutions obtained by other existing methods.

**Case 2:**

\[ \omega = \pm \frac{\sqrt{- (4\mu - \lambda^2)(a - 8\mu b^2 + 2\lambda^2 b^2)}}{4\mu - \lambda^2}, \quad a_0 = - \frac{a(2\mu + \lambda^2)}{(4\mu - \lambda^2)b}, \]  
\[ a_1 = - \frac{6a\lambda}{(4\mu - \lambda^2)b}, \quad a_2 = - \frac{6a}{(4\mu - \lambda^2)b}. \]  
(3.20)

Now, we obtain the following hyperbolic function solutions of the Klein-Gordon equation where \( \lambda^2 - 4\mu > 0 \), imposing the values of arbitrary constants and speed of the traveling wave from Eq. (3.20) into Eq. (3.12):

\[ u = \]  

\[ - \frac{a(2\mu + \lambda^2)}{(4\mu - \lambda^2)b} \left( - \frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{c_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi}{c_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi} \right) \right) \]  
(3.21)

\[ \]  

Subsequently \( c_1 \) and \( c_2 \) are arbitrary constants, one may randomly select their values. Therefore, if we select \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) respectively we obtain

\[ u = \frac{a}{2b} \left( 3 \left( \coth \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right)^2 - 1 \right) \]  
(3.22)

\[ u = \frac{a}{2b} \left( 3 \left( \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right)^2 - 1 \right) \]  
(3.23)
If $\lambda^2 - 4\mu < 0$, and by employing the values from Eq. (3.13) into Eq. (3.12), we get the following solutions

$$u = - \frac{a(2\mu + \lambda^2)}{(4\mu - \lambda^2)b} - \frac{6a\lambda}{(4\mu - \lambda^2)b} \left( - \frac{\lambda}{2} + \sqrt{\frac{4\mu - \lambda^2}{2}} \left( \frac{c_1 \sin \left( \frac{4\mu - \lambda^2}{2} \xi \right) + c_2 \cos \left( \frac{4\mu - \lambda^2}{2} \xi \right)}{c_1 \cos \left( \frac{4\mu - \lambda^2}{2} \xi \right) + c_2 \sin \left( \frac{4\mu - \lambda^2}{2} \xi \right)} \right) \right) - \frac{6a}{(4\mu - \lambda^2)b} \left( - \frac{\lambda}{2} + \sqrt{\frac{4\mu - \lambda^2}{2}} \left( \frac{c_1 \sin \left( \frac{4\mu - \lambda^2}{2} \xi \right) + c_2 \cos \left( \frac{4\mu - \lambda^2}{2} \xi \right)}{c_1 \cos \left( \frac{4\mu - \lambda^2}{2} \xi \right) + c_2 \sin \left( \frac{4\mu - \lambda^2}{2} \xi \right)} \right) \right)^2 \quad (3.24)$$

Since $c_1$ and $c_2$ are constants of integration, one may subjectively assign their values. Therefore, if we assign $c_1 = 0$ but $c_2 \neq 0$ and $c_1 \neq 0$ but $c_2 = 0$ respectively, we obtain

$$u = \frac{a}{2b} \left( 3 \left( \cot \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right)^2 - 1 \right) \quad (3.25)$$

$$u = \frac{a}{2b} \left( 3 \left( \tan \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right)^2 - 1 \right) \quad (3.26)$$

Also for Case 2, the solutions (3.22), (3.23), (3.25) and (3.26) are free of the arbitrary constant which stipulates to gain some recognized solutions obtained by other existing methods.

It is remarkably found that the traveling wave solutions of the Klein-Gordon equation in Type I, are new and have not yet been found in the previous literature.

**Type II**

In combination of the Eq. (3.1) and Eq. (3.4), the Klein-Gordon equation in (2+1)-dimension can be written as:

$$u_{tt} - k^2 (u_{xx} + u_{yy}) + au - bu^3 = 0 \quad (3.27)$$

The aforementioned equation is known as cubic nonlinear Klein-Gordon equation. In order to examine Eq. (3.27), we consider the Eq. (3.9) and Eq. (3.10). Then Eq. (3.27) converted to the next ODE:
\[(\omega^2 - 2k^2)u'' + au - bu^3 = 0 \]  \hspace{1cm} (3.28)

Comparing the highest derivative \(u''\) and nonlinear term of the highest order \(u^3\), yields \(m = 1\). Therefore, the shape of the solution (2.4) is as follows:

\[u = a_0 + a_1(G'/G)\]  \hspace{1cm} (3.29)

Here \(a_0\) and \(a_1\) are constants which require to find out for the outcomes of Eq. (3.29).

Inserting Eq. (3.29) along with Eqs. (2.5)-(2.8) into Eq. (3.28) produces a system of algebraic equations (for minimalism the equations are not displayed here). Solving these algebraic equations by means of symbolic computation software, we obtain the following values of the constants:

\[\omega = \pm \sqrt{-(\mu - \lambda^2)(a - 4\mu k^2 + \lambda^2 k^2)} \quad 4\mu - \lambda^2}, \quad a_0 = \pm \frac{a\lambda}{\sqrt{-ab(4\mu - \lambda^2)}}, \quad a_1 = \]  \hspace{1cm} (3.30)

In the case of \(\lambda^2 - 4\mu > 0\), and by means of the values assembled in (3.30) from solution Eq. (3.29), we achieve the following hyperbolic function solutions:

\[u = \pm \frac{a\lambda}{\sqrt{-ab(4\mu - \lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu - \lambda^2)}}{b(4\mu - \lambda^2)} \left( \frac{\lambda^2 - 4\mu}{2} \right)^2 \left( c_1 \sinh \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) + c_2 \cosh \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right) \right) \]  \hspace{1cm} (3.31)

Here \(c_1\) and \(c_2\) are constants of integration, thus one may randomly pick their values. Therefore, if we pick \(c_1 = 0\) but \(c_2 \neq 0\) and \(c_1 \neq 0\) but \(c_2 = 0\) respectively, we attain

\[u = \pm \frac{a\lambda}{\sqrt{-ab(4\mu - \lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu - \lambda^2)}}{b(4\mu - \lambda^2)} \left( \frac{-\lambda^2 - 4\mu}{2} \right)^2 \left( \coth \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right) \right) \]  \hspace{1cm} (3.32)

\[u = \pm \frac{a\lambda}{\sqrt{-ab(4\mu - \lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu - \lambda^2)}}{b(4\mu - \lambda^2)} \left( \frac{-\lambda^2 - 4\mu}{2} \right)^2 \left( \tanh \left( \frac{\lambda^2 - 4\mu}{2} \xi \right) \right) \right) \]  \hspace{1cm} (3.33)

With the condition \(\lambda^2 - 4\mu < 0\), and inserting the values from (3.30) into (3.29), we attain the following trigonometric function solutions
\( u = \pm \frac{a\lambda}{\sqrt{-ab(4\mu-\lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu-\lambda^2)}}{b(4\mu-\lambda^2)} \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left( \frac{c_1 \sin \frac{\sqrt{4\mu-\lambda^2}}{2} + c_2 \cos \frac{\sqrt{4\mu-\lambda^2}}{2}}{c_1 \cos \frac{\sqrt{4\mu-\lambda^2}}{2} + c_2 \sin \frac{\sqrt{4\mu-\lambda^2}}{2}} \right) \right) \)  

(3.34)

On the grounds that \( c_1 \) and \( c_2 \) are arbitrary constants, one may arbitrarily choose their values. Therefore, if we choose \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) respectively, we achieve

\[
\begin{align*}
\mathbf{u} &= \pm \frac{a\lambda}{\sqrt{-ab(4\mu-\lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu-\lambda^2)}}{b(4\mu-\lambda^2)} \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left( \cot \left( \frac{\sqrt{4\mu-\lambda^2}}{2} \xi \right) \right) \right) \\
\mathbf{u} &= \pm \frac{a\lambda}{\sqrt{-ab(4\mu-\lambda^2)}} + \left( \pm \frac{2\sqrt{-ab(4\mu-\lambda^2)}}{b(4\mu-\lambda^2)} \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu-\lambda^2}}{2} \left( \tan \left( \frac{\sqrt{4\mu-\lambda^2}}{2} \xi \right) \right) \right)
\end{align*}
\]

(3.35)  

(3.36)

Type III

In this section, we consider polynomial type nonlinear Klein-Gordon equation in (2+1)-dimension. From Eq. (3.1) and (3.5), the polynomial type Klein-Gordon equation in (2+1)-dimension form can be written as:

\[
\mathbf{u}_{tt} - k^2 (u_{xx} + u_{yy}) + au - bu^n = 0.
\]

(3.37)

In order to examine Eq. (3.37), we consider the Eq. (3.9) and (3.10). Then Eq. (3.37) converted to ODE:

\[
(a^2 - 2k^2)u'' + au - bu^n = 0.
\]

(3.38)

Equating the highest order derivatives of \( u'' \) and nonlinear term of the highest order \( u^n \), yields \( m = \frac{2}{n-1} \) which is fraction for all \( n \geq 3 \) and \( n \neq 1 \). In order to avoid the fraction power with the variable, we consider

\[
u = \frac{2}{n-1}
\]

(3.39)

From Eq. (3.38) and (3.39), we obtain the following equation where there is no fraction power with the variable
\[(\omega^2 - 2k^2)\left\{\frac{2(3-n)}{(n-1)^2} (v')^2 + \frac{2}{n-1} vv''\right\} + av^2 - bv^4 = 0. \quad (3.40)\]

Equating the highest order derivative \(vv''\) or \((v')^2\) and nonlinear term of the highest order \(v^4\) appearing in Eq. (3.39), yields \(m = 1\), then the form of the solution of Eq. (3.40) is identical to the form (3.29). Inserting Eq. (3.39) into Eq. (3.40), we obtain a polynomial of \((\mathcal{G}'/\mathcal{G})\). Setting each coefficient equal to zero, we obtain a set of algebraic equation for \(a_0, a_1\) and \(\omega\) (for simplicity we have omitted these equations) and solving them we obtain

**Case 1:**

\[
\omega = \pm \frac{\sqrt{(8\mu-2\lambda^2)(a-4\mu k^2+\lambda^2 k^2)}}{4\mu-\lambda^2}, \quad b = -\frac{4a}{(4\mu-\lambda^2)a_1}, \quad n = 3, \quad a_0 = \frac{1}{2} a_1, \quad a_1 = (3.41) \]

Considering \(\lambda^2 - 4\mu > 0\) as well as engaging the values from Eq. (3.41) into Eq. (0.40), we gain the following hyperbolic function solutions

\[
v = \frac{1}{2} a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( c_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \]

Substituting the value of \(v\) into Eq. (3.39), we obtain

\[
u = \left\{ \frac{1}{2} a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( c_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + c_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right\}^2 \quad (3.42)\]

It is observed that, \(a_1\) is an arbitrary constant in the solution of Eq. (3.42).

On account of \(c_1\) and \(c_2\) are arbitrary constants, one may arbitrarily assign their values. Therefore, if we assign \(c_1 = 0\) but \(c_2 \neq 0\) and \(c_1 \neq 0\) but \(c_2 = 0\) separately, we find

\[
u = \left\{ \frac{1}{2} a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \coth \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right) \right\}^2 \quad (3.43)\]
\[ u = \left( \frac{1}{2} a_1 + a_1 \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right)^{\frac{2}{n-1}} \]  

(3.44)

Considering the condition \( \lambda^2 - 4\mu < 0 \), and putting the values from Eq. (3.41) into Eq. (3.29), the following trigonometric solution of the Klein-Gordon equation can be achieved

\[ u = \left( \frac{1}{2} a_1 + a_1 \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{c_1 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + c_2 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi}{c_1 \cos \frac{\sqrt{4\mu - \lambda^2}}{2} \xi + c_2 \sin \frac{\sqrt{4\mu - \lambda^2}}{2} \xi} \right) \right)^{\frac{2}{n-1}} \]  

(3.45)

where \( c_1 \) and \( c_2 \) are arbitrary constants, we may randomly allocate their values. Therefore, if we allocate \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) correspondingly, we obtain

\[ u = \left( \frac{1}{2} a_1 + a_1 \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \cot \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \xi \right) \right) \right)^{\frac{2}{n-1}} \]  

(3.46)

\[ u = \left( \frac{1}{2} a_1 + a_1 \right) \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \tan \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \xi \right) \right) \right)^{\frac{2}{n-1}} \]  

(3.47)

**Case 2:**

\[ a_1 = a_1, \quad \omega = \pm \frac{\sqrt{2} \sqrt{(4\mu - \lambda^2)(2a - 4\mu k^2 + \lambda^2 k)}}{4\mu - \lambda^2}, \quad b = -\frac{a}{(4\mu - \lambda^2)a_1}, \quad n = -3, \quad a_0 = \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1. \]  

(3.48)

When \( \lambda^2 - 4\mu > 0 \), then applying the values from Eq. (3.48) together with Eq. (3.29), we accomplish the following solutions
\[ u = \left\{ \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{c_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} + c_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}}{c_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} + c_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}} \right) \right) \right\}^\frac{2}{n-1} \] (3.49)

In Eq. (3.49), \( a_1 \) is an arbitrary constant.

Since \( c_1 \) and \( c_2 \) are arbitrary constants, one may arbitrary choose their values. Therefore, if we assign \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) correspondingly, we obtain

\[ u = \left\{ \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \coth \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right\}^\frac{2}{n-1} \] (3.50)

\[ u = \left\{ \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \tanh \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right\}^\frac{2}{n-1} \] (3.51)

Again when \( \lambda^2 - 4\mu < 0 \), by means of the values from Eq. (3.48) along with Eq. (3.29), we obtain the following hyperbolic function solutions

\[ u = \left\{ \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{c_1 \sinh \frac{4\mu - \lambda^2}{2} \xi + c_2 \cosh \frac{4\mu - \lambda^2}{2} \xi}{c_1 \cosh \frac{4\mu - \lambda^2}{2} \xi + c_2 \sinh \frac{4\mu - \lambda^2}{2} \xi} \right) \right) \right\}^\frac{2}{n-1} \] (3.52)

Here \( a_1 \) is arbitrary constant.

As \( c_1 \) and \( c_2 \) are arbitrary constants, one may arbitrary choose their values. Therefore, if we choose \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) respectively we obtain

\[ u = \left\{ \left( \frac{1}{2} \lambda \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \coth \frac{\sqrt{4\mu - \lambda^2}}{2} \xi \right) \right) \right\}^\frac{2}{n-1} \] (3.53)
\[ u = \left( \frac{1}{2} \pm \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \right) a_1 + a_1 \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi \right) \right) \right) \] (3.54)

**Type IV**

In this case, using (3.1) and (3.6), the Klein-Gordon equation in (2+1)-dimension can be written as:

\[ u_{tt} - k^2 (u_{xx} + u_{yy}) + au - bu^n + cu^{2n-1} = 0. \] (3.55)

Therefore, in order to examine Eq. (3.55), we consider the Eq. (3.9) and Eq. (3.10). Then Eq. (3.55) converted into an ordinary differential equation (ODE):

\[ (\omega^2 - 2k^2)u'' + au - bu^n + cu^{2n-1} = 0 \] (3.56)

Harmonizing the highest order derivatives of \( u'' \) and nonlinear term of the highest order \( u^{2n-1} \), yields \( m = \frac{1}{n-1} \) which is fraction for all \( n > 1 \) and \( n \neq 1 \). In order to avoid the fraction power with the variable, we consider

\[ u = v^{\frac{1}{n-1}} \] (3.57)

From Eq. (3.55) and (3.56), we obtain the following equation where there is no fraction power with the variable

\[ (\omega^2 - 2k^2) \left( \frac{(2-n)}{(n-1)^2} (v')^2 + \frac{1}{n-1} vv'' \right) + av^2 - bv^3 + cv^4 = 0 \] (3.58)

Equating the highest order derivatives of \( vv'' \) (or \( (v')^2 \)) and nonlinear term of the highest order \( v^4 \), yields \( m = 1 \) and thus the form of the solution of Eq. (3.58) is same as the form of (3.29).

Inserting Eq. (3.29) into Eq. (3.58), we obtain a polynomial of \((G'/G)\). Equating each coefficient equal to zero, we get a set of algebraic equation for \( a_0 \), \( a_1 \) and \( \omega \) (for minimalism we have omitted these equations) and solving them, we find

\[ \omega = \pm \sqrt{(4n-2\lambda)(a - 2an^2 + 2\mu k^2 - 2\lambda k^2 - \lambda^2)} \quad c = \frac{n b^2}{a(n+1)^2} \quad a_0 = \frac{1}{2b} \left( a n + 1 \pm \frac{\lambda}{b \sqrt{\lambda^2 - 4\mu}} \right) \]

\[ a_1 = \pm \frac{a(1+n)}{b \sqrt{\lambda^2 - 4\mu}}. \] (3.59)
When $\lambda^2 - 4\mu > 0$, adopting the values from Eq. (3.59) into Eq. (3.29), we acquire the following hyperbolic function solutions

$$u = \left\{ \frac{1}{2b} \left( a \left( n + 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{n\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \right) \right\}$$

The coefficients $c_1$ and $c_2$ are arbitrary constants, one may arbitrary choose their values. Therefore, if we consider $c_1 = 0$ but $c_2 \neq 0$ and $c_1 \neq 0$ but $c_2 = 0$ individually we get

$$u = \left\{ \frac{1}{2b} \left( a \left( n + 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{n\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \right) \right\}$$

When $\lambda^2 - 4\mu < 0$, applying the values from Eq. (3.59) into Eq. (3.29), we find the following trigonometric function solution

$$u = \left\{ \frac{1}{2b} \left( a \left( n + 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{n\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \right) \right\}$$

Where $c_1$ and $c_2$ are arbitrary constants, one may arbitrary choose their values. Therefore, if we choose $c_1 = 0$ but $c_2 \neq 0$ and $c_1 \neq 0$ but $c_2 = 0$ individually we obtain:
\[ u = \left( \frac{1}{2b} \right) \left( a \left( n + 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{n\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \right) \]

\[ \pm \frac{a(1 + n)}{b\sqrt{\lambda^2 - 4\mu}} \left( -\frac{\lambda}{2} \right) \]

\[ + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \cot \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \xi \right) \right)^{\frac{1}{n-1}} \]

\[ u = \left( \frac{1}{2b} \right) \left( a \left( n + 1 \pm \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}} \pm \frac{n\lambda}{\sqrt{\lambda^2 - 4\mu}} \right) \right) \]

\[ \pm \frac{a(1 + n)}{b\sqrt{\lambda^2 - 4\mu}} \left( -\frac{\lambda}{2} \right) \]

\[ + \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \tan \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \xi \right) \right)^{\frac{1}{n-1}} \]

**Type V**

By using Eq. (3.1) and (3.7), we have the nonlinear polynomial Klein-Gordon equation in (2+1)-dimension as:

\[ u_{tt} - k^2(u_{xx} + u_{yy}) + au - bu^{1-n} + cu^{n+1} = 0 \]  \( (3.66) \)

For solving the Eq. (3.66), we will consider the Eq. (3.9) and Eq. (3.10). Then Eq. (3.66) becomes ordinary differential equation (ODE):

\[ (\omega^2 - 2k^2)u'' + au - bu^{1-n} + cu^{n+1} = 0 \]  \( (3.67) \)

Equalizing the highest order derivatives of linear term \( u'' \) and nonlinear term of the highest order \( u^{n+1} \) in Eq. (0.67), yields \( m = \frac{2}{n} \) which is fraction for all \( n > 2 \). In order to avoid the fraction power with the variable, we consider
From Eq. (3.67) and Eq. (3.68), we obtain the following equation where there is no fraction power with the variable

\[ (\omega^2 - 2k^2) \left( \frac{2(2-n)}{n^2} (v')^2 + \frac{2}{n} v^2 \right) + av^2 - b + cv^4 = 0 \]  

(3.69)

Balancing the highest order derivatives of \( v \) (or \( v' \)) and nonlinear term of the highest order \( v^4 \), yields \( m = 1 \), then the form of the solution of Eq. (3.69) is similar to the form (3.29).

Inserting Eq. (3.29) into Eq. (3.69), we obtain a polynomial of \( (G'/G) \). Putting each coefficient equal to zero, we acquire a set of algebraic equation for \( a_0, a_1 \) and \( \omega \) (for minimalism we have omitted these equations) and solving them we get,

\[
\omega = \pm \frac{1}{2} \sqrt{\frac{(\lambda^2 - 4\mu)(an^2 - 16\mu k^2 + 4\lambda^2 k^2)}{\lambda^2 - 4\mu}}, \quad c = \frac{1}{16} \frac{a^2 (n^2 - 4)}{b}, \quad a_0 = \]

(3.70)

\[
a_1 = \pm \frac{4\lambda^2 - a(\lambda^2 - 4\mu)(n - 2)b}{a(n\lambda^2 - 2\lambda^2 - 4\mu n + 8\mu)}
\]

At a stage when \( \lambda^2 - 4\mu > 0 \), then using values from Eq. (3.70) and Eq. (3.29), we obtain the following hyperbolic function solutions

\[
u = \pm \frac{4\lambda^2 - a(\lambda^2 - 4\mu)(n - 2)b}{a(n\lambda^2 - 2\lambda^2 - 4\mu n + 8\mu)} \left( -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{c_1 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2} + c_2 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2}}{c_1 \cosh \frac{\sqrt{\lambda^2 - 4\mu}}{2} + c_2 \sinh \frac{\sqrt{\lambda^2 - 4\mu}}{2}} \right) \right)^{\frac{1}{2}}
\]

(3.71)

As \( c_1 \) and \( c_2 \) are arbitrary constants, one may arbitrary choose their values. Therefore, if we choose \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) individually, we obtain

\[ u = \left( \pm \frac{2a\sqrt{\lambda^2-4\mu}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \pm \frac{4\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \right) \left( \frac{-\mu}{2} \right) + \frac{\lambda}{2} \]  \hspace{1cm} (3.72)

\[ u = \left( \pm \frac{2a\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \pm \frac{4\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \right) \left( \frac{-\mu}{2} \right) + \frac{\lambda}{2} \]  \hspace{1cm} (3.73)

\[ \sqrt{\frac{\lambda^2-4\mu}{2}} \left( \coth \frac{\sqrt{\lambda^2-4\mu}}{2} \xi \right) \right) \left( \pm \frac{2a\sqrt{\lambda^2-4\mu}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \right) \left( \frac{-\mu}{2} \right) + \frac{\lambda}{2} \]  \hspace{1cm} (3.74)

When \( \lambda^2 - 4\mu < 0 \), then using the values from Eq. (0.70) into Eq. (0.29), we obtain the following trigonometric function solutions

\[ u = \left( \pm \frac{2a\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \pm \frac{4\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \right) \left( \frac{-\mu}{2} \right) + \frac{\lambda}{2} \]  \hspace{1cm} (3.75)

Since \( c_1 \) and \( c_2 \) are arbitrary constants, one may randomly assign their values. Therefore, if we choose \( c_1 = 0 \) but \( c_2 \neq 0 \) and \( c_1 \neq 0 \) but \( c_2 = 0 \) reciprocally, we obtain

\[ u = \left( \pm \frac{2a\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \pm \frac{4\sqrt{-a(\lambda^2-4\mu)(n-2)b}}{a(n\lambda^2-2\lambda^2-4\mu+n+8\mu)} \right) \left( \frac{-\mu}{2} \right) + \frac{\lambda}{2} \]  \hspace{1cm} (3.76)

III. Conclusion

The \((G'/G)\)-expansion method has been successfully implemented to investigate the general traveling wave solutions of the Klein-Gordon equation where five types of nonlinearity were considered. We achieved new exact traveling wave solutions in the form of hyperbolic and trigonometric functions. Also, we gained sets of solutions by assigning one of the arbitrary constants equal to zero. The obtained solutions are involving arbitrary constants which give generalized wave solutions of
the Klein-Gordon equation. If we set particular values of the unknown constants, we achieved different known solutions which validate the obtained solutions. The readiness of computer system, for example, Maple helps us to simplify the tedious calculations in this article. The solutions accomplished in this article are promising since there are some free parameters in the solutions. The solutions can be used to explain the physical insight to the problems considered for example quantum field theory, nonlinear optics where the Klein-Gordon equation has the nonlinearity of quadratic, cubic and polynomials. The efficiency of this method is precise, reliable and robust, which can be adapted to other NLEEs arises in the field of physics, applied mathematics and engineering.

Reference


