GLIVENKO CONGRUENCE ON A 0-DISTRIBUTIVE MEET SEMILATTICE

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Abstract:

In this paper the author studies the Glivenko congruence \( R \) in a 0-distributive meet semilattice. It is proved that a meet semilattice \( S \) with 0 is 0-distributive if and only if the quotient semilattice \( \frac{S}{R} \) is distributive. Hence \( S \) is 0-distributive if and only if \((0]\) is the kernel of some homomorphism of \( S \) onto a distributive meet semilattice with 0.

Key words and phrases: Glivenko congruence, 0-distributive semilattice, distributive meet semilattice.

Introduction:

J.C. Varlet [7] first introduced the concept of 0-distributive lattices. Then many authors including [1,2,3,5] studied them for lattices and semilattices. By [2], a meet semilattice \( S \) with 0 is called 0-distributive if for all \( a,b,c \in S \) with \( a \land b = 0 = a \land c \) imply \( a \land d = 0 \) for some \( d \geq b,c \). A meet semilattice \( S \) is called directed above if for all \( a,b \in S \), there exists \( c \in S \) such that \( c \geq a,b \). We know that all modular and distributive semilattices have the directed above property. Moreover, [3] have shown that every 0-distributive meet semilattice is directed above.

Let \( S \) be a meet semilattice with 0. For a non-empty subset \( A \) of \( S \), we define \( A^{\perp} = \{ x \in S \mid x \land a = 0 \text{ for all } a \in A \} \). This is clearly a down set, but we can not prove that this is an ideal even in a distributive meet semilattice, when \( A \) is infinite.

By [2,3] we know that, for any \( a \in S \), \( \{a\}^{\perp} \) is an ideal if and only if \( S \) is 0-distributive.

We define a relation \( R \) on a meet semilattice \( S \) by \( a \equiv b(R) \) if and only if \( \{a\}^{\perp} = \{b\}^{\perp} \). In other words, \( a \equiv b(R) \) is equivalent to “for each \( x \in S \), \( a \land x = 0 \) if and only if \( b \land x = 0 \)”. 
We will show below that this is a congruence on the meet semilattice $S$. We call it Glivenko congruence. In this paper we establish some results on this congruence in a meet semilattice.

We start with the following result which is due to [3]. We include its proof for the convenience of the reader.

**Lemma 1:** Let $S$ be a meet-semilattice with $0$. Again let $A, B \subseteq S$ and $a, b \in S$ then we have the followings:

- (i) If $A \cap B = (0)$, then $B \subseteq A^\perp$
- (ii) $A \cap A^\perp = (0)$,
- (iii) $A \subseteq B$ imply that $B^\perp \subseteq A^\perp$
- (iv) If $a \leq b$ imply that $\{b\}^\perp \subseteq \{a\}^\perp$ and $\{a\}^{\perp \perp} \subseteq \{b\}^{\perp \perp}$
- (v) $\{a\}^\perp \cap \{a\}^{\perp \perp} = (0)$
- (vi) $\{a \wedge b\}^{\perp \perp} = \{a\}^{\perp \perp} \cap \{b\}^{\perp \perp}$
- (vii) $A \subseteq A^{\perp \perp}$
- (viii) $A^{\perp \perp \perp} = A^\perp$

**Proof:** (i) Let $b \in B$. Then $a \wedge b = 0$ for all $a \in A$, as $A \cap B = (0)$. Thus $b \in A^\perp$. Hence $B \subseteq A^\perp$.

(ii) Let $x \in A \cap A^\perp$.

\[ x \in A \text{ and } x \wedge a = 0 \text{ for all } a \in A \]

\[ x \wedge x = 0 \]

\[ x = 0 \]

(iii) Let $A \subseteq B$

\[ \therefore A \cap B^\perp \subseteq B \cap B^\perp = (0) \]

\[ \Rightarrow A \cap B^\perp = (0) \]

So, by (i), $B^\perp \subseteq A^\perp$.

(iv) Let $x \in \{b\}^\perp$. Then $b \wedge x = 0$ for some $x \in S$. Since $a \leq b$, then we have $a \wedge x = 0$ for some $x \in S$, which imply that $x \in \{a\}^\perp$.

Hence, $\{b\}^\perp \subseteq \{a\}^\perp$.

Now let $x \in \{a\}^{\perp \perp}$. Then $y \wedge x = 0$ for all $y \in \{a\}^\perp$, which implies that $y \wedge x = 0$ for all $y \in \{b\}^\perp$ as $\{b\}^\perp \subseteq \{a\}^\perp$. Thus $x \in \{b\}^{\perp \perp}$.

Hence,
\( \{a\}^\perp \subseteq \{b\}^\perp \).

(v) Let \( x \in \{a\}^\perp \cap \{a\}^\perp \). Then \( x \in \{a\}^\perp \) and \( x \in \{a\}^\perp \) which implies that \( x \land a = 0 \) and \( x \land y = 0 \) for all \( y \in \{a\}^\perp \). Thus \( x \land x = 0 \).
Hence
\[ \{a\}^\perp \cap \{a\}^\perp = \{0\} \]

(vi) Let \( x \in \{a\}^\perp \cap \{b\}^\perp \) and \( y \in \{a \land b\}^\perp \). Then we get \( (y \land a) \land b = 0 \), which implies that \( (y \land a) \in \{b\}^\perp \). Since \( x \in \{b\}^\perp \), we get \( (x \land y) \land a = 0 \).
Hence \( x \land y \in \{a\}^\perp \). Since \( x \in \{a\}^\perp \), we get \( x \land y \in \{a\}^\perp \). Thus \( x \land y = 0 \) for all \( y \in \{a \land b\}^\perp \). Therefore \( x \in \{a \land b\}^\perp \). Thus \( \{a\}^\perp \cap \{b\}^\perp \subseteq \{a \land b\}^\perp \).

Conversely we can write that \( a \land b \leq a \), which implies by (i) \( \{a \land b\}^\perp \subseteq \{a\}^\perp \). Similarly \( \{a \land b\}^\perp \subseteq \{b\}^\perp \). Therefore we have, \( \{a \land b\}^\perp \subseteq \{a\}^\perp \cap \{b\}^\perp \).

(vii) Let \( x \in A \), consider any \( r \in A^\perp \), then we get \( x \land a = 0 \) for all \( a \in A \) which implies that \( r \land x = 0 \). Since \( x \land r = 0 \) for all \( r \in A^\perp \). Thus \( x \in A^\perp \). Hence \( A \subseteq A^\perp \).

(viii) Since by (vii) \( A \subseteq A^\perp \). So by (iii) \( (A^\perp)^\perp \subseteq A^\perp \).
Hence \( A^\perp \subseteq A^\perp \). Since by (vii) \( A^\perp \subseteq (A^\perp)^\perp = A^\perp \). Therefore we have \( A^\perp = A^\perp \).
Hence the proof is completed. \( \square \)

**Theorem 2:** \( R \) is a meet congruence on \( S \).

**Proof:** It is clearly an equivalent relation.

Let \( a \equiv b(R) \) and \( t \in S \)

Then \( \{a\}^\perp = \{b\}^\perp \), so by using Lemma 1, we have \( \{a \land t\}^\perp = \{a \land t\}^\perp \)
\[ = \{\{a\}^\perp \land \{t\}^\perp \}^\perp \]

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This implies \( a \land t \equiv b \land t(R) \), and so \( R \) is a meet congruence on \( S \).

A meet semilattice \( S \) with 0 is weakly complemented if for any pair of distinct elements \( a \), \( b \) of \( S \), there exists an element \( c \) disjoint from one of these elements but not from the other. In particular, if \( a < b \), then there exists \( c \in S \) such that \( a \land c = 0 \) but \( b \land c \neq 0 \).

**Theorem 3:** If \( S \) is weakly complemented, then \( R \) is an equality relation.

**Proof:** Suppose \( a, b \in S \) with \( a \neq b \). Since \( S \) is weakly complemented, so there exist \( x \in S \), \( a \land x = 0 \) but \( b \land x \neq 0 \). This implies \( (a, b) \notin R \). Hence \( R \) is an equality relation.

**Theorem 4:** For any meet semilattice \( S \), \( S \) is also a meet semilattice. Moreover \( S \) is directed above if and only if \( S \) is directed above.

**Proof:** For \( [a], [b] \in \frac{S}{R} \), define \( [a]R \land [b]R = [a \land b]R \). Thus \( \frac{S}{R} \) is a meet semilattice.

Now let \( a, b \in S \). If \( S \) is directed above, then there exists \( d \geq a, b \).

Now, \( [a]R \land [d]R = [a \land d]R = [a]R \) and \( [b]R \land [d]R = [b \land d]R = [b]R \)

Implies \( [d]R \geq [a]R, [b]R \). Thus, \( \frac{S}{R} \) is also directed above.

Conversely suppose \( \frac{S}{R} \) is directed above. Let \( a, b \in S \)

Then \( [a], [b] \in \frac{S}{R} \). Since \( \frac{S}{R} \) is directed above, so there exists \( C \in \frac{S}{R} \) such that \( C \geq [a]R, [b]R \). Then there exists \( d \in C \), such that \( [d] = C \) and \( d \geq a, b \). So \( S \) is directed above.

A meet semilattice \( S \) is called a *distributive semilattice* if \( w \geq a \land b \) implies that there exist \( x \geq a \), \( y \geq b \) in \( S \) such that \( w = x \land y \).
Following result gives some characterizations of distributive meet semilattices which are due to [4, Theorem 1.1.6], also see [6].

**Lemma 5:** For a meet semilattice $S$, the following conditions are equivalent.

i) $S$ is distributive.

ii) $w \geq a \land b$ implies that there exists $y \in S$ such that $y \geq b$, $y \geq w$ and $y \land a = a \land w$.

iii) $a \land b = b \land c$ implies that there exists $y \in S$ such that $y \geq b$, $y \geq c$ and $y \land a = a \land c$.

**Theorem 6:** For any meet semilattice $S$, the quotient meet semilattice $\frac{S}{R}$ is weakly complemented. Furthermore, $S$ is $0$-distributive if and only if $\frac{S}{R}$ is distributive.

**Proof:** First part: For any meet semilattice $S$, when $A < B$ in $\frac{S}{R}$, there exists $a \in A$ and $b \in B$ such that $a < b$, and by the definition of $R$, there is an element $c$ such that $0 = \bot c a$ and $0 \neq \bot b c$. Since the minimum class of $\frac{S}{R}$ has the only element 0, the class $C$ of $c$ satisfies $A \land C = [0]$ and $C \land B \neq [0]$. Therefore, $\frac{S}{R}$ is weakly complemented.

For second part: Let $S$ be 0-distributive. Suppose $B \geq A \land C$ in $\frac{S}{R}$. So there exists $b \in B$, $a \in A$, $c \in C$ such that $b \geq a \land c$ and $B = [b]R$, $A = [a]R$, $C = [c]R$. Suppose $a \land b \land x = 0$. Then $a \land c \land x = 0$. Since $S$ is 0-distributive, so there exists $d \geq b, c$ such that $a \land d \land x = 0$. On the other hand, for any $d \geq b, c$, $a \land d \land x = 0$ implies $a \land d \land x \land b = a \land b \land x = 0$. Therefore, $a \land b \equiv a \land d(R)$ for some $d \geq b, c$. In other words, $A \land B = A \land D$ where $D = [d] \geq B, C$.

Therefore by [4, Theorem 1.1.6 (ii)], $\frac{S}{R}$ is distributive.

Conversely, suppose $\frac{S}{R}$ is distributive. Let $a, b, c \in S$ with $a \land b = a \land c = 0$. Then $[a] \land [b] = [a] \land [c] = [0]R$. Since $[0]$ contains only the element 0, so $A \land B = A \land C = 0$, where $A = [a]$, $B = [b]$, $C = [c]$. Then $B \geq A \land C$. Since $\frac{S}{R}$ is distributive, so $B = A_i \land C_1$ for some $A_i \geq A$, $C_1 \geq C$.

Moreover, $B = A_i \land C_1$ implies $C_1 \geq B$. Thus $0 = A \land B = A \land A_i \land C = A \land C_1$. 

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Now \( C_1 \geq B, C \) implies \( C_1 = [d]R \) for some \( d \geq b, c \). Therefore, \( a \land d = 0 \) for some \( d \geq b, c \) and so \( S \) is 0-distributive.

We conclude the paper with the following result.

**Theorem 7:** Let \( S \) be a meet semilattice. Then the following conditions are equivalent

(i) \( S \) is 0-distributive.

(ii) \( (0) \) is the kernel of some homomorphism of \( S \) onto a distributive semilattice with 0.

(iii) \( (0) \) is the kernel of a homomorphism of \( S \) onto a 0-distributive semilattice.

**Proof:**

(i) \( \Rightarrow \) (ii). Suppose \( S \) is 0-distributive. Then by Theorem 1, the binary relation \( R \) defined by \( x \equiv y(R) \iff (x)^+ = (y)^+ \) is a congruence on \( S \). Moreover by Theorem 5, \( S/R \) is a distributive meet semilattice. Clearly the map \( a \mapsto [a]R \) is a homomorphism. Now let \( a \equiv 0(R) \). Then \( 0 \land a = 0 \) implies \( a = a \land a = 0 \). Here \( [0]R \) contains only 0 of \( S \). That is, \( (0) \) is a complete congruence class modulo \( R \).

(ii) \( \Rightarrow \) (iii) is obvious as every distributive semilattice with 0 is 0-distributive.

(iii) \( \Rightarrow \) (i). Let \( \Theta \) be a congruence on \( S \) for which \( (0) \) is the zero element of the 0-distributive semilattice \( S/\Theta \). Then \( x \land y = 0 = x \land z \) imply

\[
[x]\Theta \land [y]\Theta = [x \land y]\Theta = [x \land z]\Theta = [x]\Theta \land [z]\Theta .
\]

Thus, \( [x]\Theta \land [y]\Theta = (0) = [x]\Theta \land [z]\Theta \). Hence by the 0-distributivity of \( S/\Theta \),

\[
[x]\Theta \land [d]\Theta = (0) , \text{ for some } [d]\Theta \geq [y]\Theta, [z]\Theta .
\]

This implies \( x \land d \in (0) \) and so \( x \land d = 0 \), where \( d \geq y, z \). Therefore, \( S \) is 0-distributive. \( \Box \)
References


