Families of exact traveling wave solutions to the space time fractional modified KdV equation and the fractional Kolmogorov-Petrovskii-Piskunov equation

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Abstract:

The space time fractional modified KdV equation and fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation models the unidirectional and bidirectional waves on shallow water surfaces, long internal wave in a density-stratified ocean, ion acoustic waves in plasma, acoustic waves on a crystal lattice. The fractional derivatives are defined in the modified Riemann-Liouville sense. In this article, we obtain exact solution of these equations by means of the recently established two variables \((G'/G, 1/G)\)-expansion method. The solutions are obtained in the form of hyperbolic, trigonometric and rational functions involving parameters. When the parameters are assigned particular values, the solitary wave solutions are generated from the traveling wave solutions. The method indicates that it is easy to implement, computationally attractive and is the general form of the original \((G'/G)\)-expansion method.

Keywords: Exact solution; fractional modified KdV equation; Kolmogorov-Petrovskii-Piskunov equation; modified Remann-Liouville derivative; traveling wave solution; solitary wave solution.

I. Introduction

In the last few decades many researchers demonstrate that the integrals and derivatives of arbitrary order are very compatible for the explanation of properties
of diverse materials, as for example polymers. It has been shown that the fractional order models are further adequate than the integer order models. The necessity of fractional derivatives becomes apparent in modeling of electrical and mechanical properties of real materials, as well as in the description of rheological properties of rocks. Therefore, the topic fractional differential equation is a field of mathematical study that grows out of the fundamental definition of integral and differential operators. The nonlinear fractional differential equations (NLFDDEs) also play a significant role in optical fiber, signal processing, control theory, earthquake, fluid mechanics, plasma physics, relativity, chemical physics, solid state physics, geochemistry, biomechanics, ecology, gas-dynamics, bio-physics and so on. The exact solutions of NLFDDEs help us to provide information about the structure of complex phenomena. In the last two decades exact solutions of NLFPDEs have been analyzed by many researchers [XXVII, XXIX, XXII] who are interested in nonlinear physical and engineering events. They established several methods, such as the Adomian’s decomposition method [XI, X, XXX], the variational iteration method [XLI, XX, XL, XVI, XXXI], the homotopy analysis method [XXXVII, IV], the homotopy perturbation method [XIV, XVIII], the differential transformation method [XXVIII, XII], the fractional sub-equation method [XXV, XVII, XLV], extended auxiliary equation method [XXXV], the multiple scale method [XXXVI], the \((G'/G)\)-expansion method [XXXIX, XLVI, V, VI, XV, XXIII, XLII, I, II], the first integral method [XXVI, VIII], the exp-function method [XLIV, VII, III], the two variables\((G'/G, 1/G)\)-expansion method [XXXVIII, XXIV, XLIII], the direct algebraic method [XXXII, XXXIII, XXXIV] etc.

Recently, Li et al. [XXIV] established the two variables\((G'/G, 1/G)\)-expansion method to extract the exact solutions of the NLFDDEs which is further convenient, effective and easy to compute. It is very powerful mathematical tools for finding exact solutions of NLFDDEs. By means of this method [XXXVIII], exact solutions of the fractional generalized reaction Duffing model and the density dependent fractional diffusion reaction equation and some others equations were investigated.

It can be mentioned that, the two variables \((G'/G, 1/G)\)-expansion method is the general form of \((G'/G)\)-expansion method. The main idea of this method is that the solutions of NLFDDEs are presented as a polynomial in two variables \((G'/G)\) and \((1/G)\). where\(G = G(\xi)\) satisfies the second order ODE \(G''(\xi) + \lambda G(\xi) = \mu\), where \(\lambda\) and \(\mu\) are constants.

The objective of this article is to establish further general and new close form exact traveling wave solution of the space time fractional modified Korteweg-de Vries (KdV) equation and the space time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation by means of the two variables\((G'/G, 1/G)\)-expansion method. The space time fractional modified KdV equation is as follows:
\[ D_t^\alpha u + ku^2 D_x^\alpha u - \tau D_x^{3\alpha} u = 0, \quad 0 < \alpha \leq 1, \]  
(1.1)

where \( k \) and \( \tau \) are nonzero constants.

The space time fractional KPP equation is

\[ D_t^\alpha u - D_x^{2\alpha} u + \mu_1 u + \gamma u^2 + \delta u^3 = 0, \quad 0 < \alpha \leq 1, \]  
(1.2)

where \( \mu_1, \gamma \) and \( \delta \) are arbitrary constants.

The rest of this article is allocated as follows: In section 2 and 3 we describe the Jumarie’s modified Riemann-Liouville derivative and the two variable \((G'/G, 1/G)\)-expansion method, in section 4, we establish the exact solution of the space time fractional modified Korteweg-de Vries (KdV) equation and the space time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation by the proposed method and in the last section the conclusions are given.

**II. Jumarie modified Riemann-Liouville derivative**

There are several types of the generalization to the expression of differentiation to fractional order as for example Riemann-Liouville definition and Caputo definition [IX, XXI]. Jumarie established a modified Riemann-Liouville derivative and using this kind of fractional derivative we reduce fractional differential equation into integer order differential equation by variable transformation. Suppose that \( f: \mathbb{R} \to \mathbb{R}, x \to f(x) \) denote a continuous (but not necessarily differentiable) function. The Jumarie modified Riemann-Liouville derivative of order \( \alpha \) is defined as:

\[
D_x^\alpha f(x) = \begin{cases} 
\frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} [f(\xi) - f(0)]d\xi, & \alpha < 0 \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} [f(\xi) - f(0)]d\xi, & 0 < \alpha < 1 \\
(f^{(n)}(x))^{(\alpha-n)}, & n \leq \alpha \leq n+1, n \geq 1
\end{cases}
\]  
(2.1)

Some fundamental properties for the modified Riemann-Liouville derivative are as follows:

\[
D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{(\gamma-\alpha)}, \quad \gamma > 0
\]  
(2.2)

\[
D_t^\alpha (af(t) + bg(t)) = aD_t^\alpha f(t) + bD_t^\alpha g(t),
\]  
(2.3)

where \( a \) and \( b \) are constants, and

\[
D_t^\alpha c = 0, \text{ here } c \text{ is constant.}
\]  
(2.4)
which are the direct consequence of
\[ d^\alpha x(t) = \Gamma(1 + \alpha)dx(t) \quad (2.5) \]

III The two variable \((G'/G, 1/G)\)-expansion method

Let us consider the second order equation
\[ G''(\xi) + \lambda G(\xi) = \mu \quad (3.1) \]
and we consider the following relations
\[ \phi = G'/G, \ \psi = 1/G \quad (3.2) \]
Thus, we obtain
\[ \phi' = -\phi^2 + \mu \psi - \lambda, \ \psi' = -\phi \psi \quad (3.3) \]
The solution of the above equation \((3.1)\) depends on \(\lambda\) for which its sign as
\(\lambda < 0, \lambda > 0\) and \(\lambda = 0\)

**Remark1:** If \(\lambda < 0\), the general solution of equation \((3.1)\) is:
\[ G(\xi) = A_1 \sinh(\sqrt{-\lambda} \ \xi) + A_2 \cosh(\sqrt{-\lambda} \ \xi) + \frac{\mu}{\lambda} \quad (3.4) \]
where \(A_1\) and \(A_2\) are arbitrary constants. Consequently, we obtain
\[ \psi^2 = \frac{-\lambda}{\lambda^2 + \mu^2} (\phi^2 - 2\mu \psi + \lambda), \quad (3.5) \]
where \(\sigma = A_1^2 - A_2^2\).

**Remark2:** If \(\lambda > 0\), the general solution of equation \((3.1)\) is:
\[ G(\xi) = A_1 \sin(\sqrt{\lambda} \ \xi) + A_2 \cos(\sqrt{\lambda} \ \xi) + \frac{\mu}{\lambda} \quad (3.6) \]
where \(A_1\) and \(A_2\) are arbitrary constants. As a result, we obtain
\[ \psi^2 = \frac{\lambda}{\lambda^2 - \mu^2} (\phi^2 - 2\mu \psi + \lambda), \quad (3.7) \]
where \(\sigma = A_1^2 + A_2^2\).

**Remark3:** If \(\lambda = 0\), the general solution of equation \((3.1)\) is:
\[ G(\xi) = \frac{\mu}{2} \xi^2 + A_1 \xi + A_2 \quad (3.8) \]
where $A_1$ and $A_2$ are arbitrary constants. Accordingly, we attain

\[
\psi^2 = \frac{1}{A_1^2 - 2 \mu A_2} \left( \phi^2 - 2 \mu \psi \right)
\]  

(3.9)

We consider the general nonlinear fractional differential equation (NLFDE) of the type

\[
P(u, D_x^\alpha u, D_x^\beta u, D_x^\gamma D_t^\alpha u, D_x^\delta D_t^\beta u, D_x^\rho D_t^\gamma D_t^\delta u, \ldots \ldots ) = 0,
\]

\[0 < \alpha \leq 1, \ 0 < \beta \leq 1.
\]  

(3.10)

where $u$ is an unknown function of spatial derivative $x$ and temporal derivative $t$ and $P$ is a polynomial of $u$ and its partial fractional derivatives wherein the maximum number of derivatives and the nonlinear terms are involved.

Step 1: Consider the traveling wave transformation

\[
\xi = \frac{x^\beta}{\Gamma(1+\beta)} + \frac{\omega x^\alpha}{\Gamma(1+\alpha)} u(x, t) = u(\xi).
\]  

(3.11)

where $\omega$ is a nonzero arbitrary constant named the velocity of the wave.

Using this transformation, we can rewrite the equation (3.10) as:

\[
Q(u, u', u'', u''', \ldots \ldots ) = 0.
\]  

(3.12)

where prime stands for the derivative of $u$ with respect to $\xi$.

Step 2: Assume that the solution of equation (3.12) can be written as a polynomial in two variables $\phi$ and $\psi$ as follows:

\[
\psi(\xi) = \sum_{i=0}^{N_b} a_i \phi^i + \sum_{i=1}^{N_c} b_i \phi^{i-1} \psi
\]  

(3.13)

wherein $a_i$ ($i = 0, 1, 2, \ldots$) and $b_i$ ($i = 0, 1, 2, \ldots$) are constants to be determined afterward.

Step 3: The homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (3.12) determined the positive integer $N$.

Step 4: Substitute (3.13) into (3.12) together with (3.3) and (3.5), the equation (3.12) can be reduced into a polynomial in $\phi$ and $\psi$, wherein the degree of $u$ is not greater than one. Equating the coefficients of this polynomial of like power to zero give a system of algebraic equations which can be solved by using the computer algebra, like Maple or Mathematica yields the values of $a_i, b_i, \mu, A_1, A_2$ and $\lambda$ where $\lambda < 0$, which provide hyperbolic function solutions.

Step 5: Similarly, we examine the values of $a_i, b_i, \mu, A_1, A_2$ and $\lambda$ when $\lambda > 0$ and $\lambda = 0$, yield the trigonometric and rational function solutions respectively.
### IV Formation of exact solutions

4.1: In this subsection, we investigate some further general and new closed form wave solutions to the space time fractional modified Korteweg-de Vries (mKdV) equation (1.1) by means of the two variables $\left( G'/G, 1/G \right)$-expansion method.

For the mKdV equation (1.1), we introduce the following transformation:

$$
\xi = \frac{x^\alpha}{r^{1+\alpha}} + \frac{\omega t^\alpha}{r^{1+\alpha}} u(x, t) = u(\xi),
$$

where $\omega$ is the speed of the traveling wave. Using the transformation (4.1), the mKdV equation (1.1) reduces to the following integer order ordinary differential equation (ODE):

$$
\omega u' + ku^2 u' + \tau u''' = 0.
$$

Integrating equation (4.2) with zero constant, we obtain

$$
\omega u + \frac{ku^3}{3} + \tau u'' = 0,
$$

where the integral constant is regarded as zero.

Balancing the highest order derivative $u''$ and the nonlinear term $u^3$, yields $N = 1$. So, the solution of (4.2) reduces to the following shape:

$$
u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi),$$

where $a_0, a_1$ and $b_1$ are constants to be evaluated.

Case 1: For $\lambda < 0$, inserting equation (4.4) into equation (4.3) alongside with equation (3.3) and equation (3.5) yields a set of algebraic equations:

$$
\phi^0: -2k\mu b_1^3 \lambda^3 + 2k\mu^2 a_0^3 \lambda^2 \sigma + 3\omega a_0 \mu + 6\omega a_0 \lambda^2 \sigma - 3ka_2 b_1^2 \lambda^5 \sigma + 3\tau b_1 \lambda^4 \mu \sigma + ka_2^2 \lambda^6 \sigma^2 + k\mu^3 a_0^2 - 3k\mu^2 a_0 b_2^2 \lambda^2 + 3\omega a_0 \lambda^4 \sigma^2 + 3\tau b_2 \lambda^2 \mu^2 = 0
$$

$$
\phi^1: + 6\tau a_2 \lambda^5 \sigma + 12\tau a_2 \lambda^3 \sigma^2 + 3\omega a_2 \lambda^4 \sigma^2 + 3\omega a_2 \lambda^6 \sigma + 3k\mu^2 a_0 b_1^2 \lambda^2 + 6k\mu^2 a_0^2 a_1 \lambda^2 \sigma - 3ka_2 b_1^2 \lambda^4 \sigma + 3k\mu^4 a_0^2 a_1 + 3ka_2^2 a_1 \lambda^4 \sigma^2 + 6\omega a_2 \lambda^2 \mu \sigma + 6\tau a_2 \lambda^4 \mu^2 = 0
$$

$$
\phi^2: 3\tau b_1 \lambda^3 \mu \sigma + 6\mu^2 a_0 a_2^2 \lambda^2 \sigma + 3\tau b_1 \lambda \mu^3 - 2k\mu b_1^3 \lambda^2 + 3ka_2^2 \lambda^4 \sigma^2 - 3ka_2 b_1^2 \lambda^3 \sigma - 3k\mu^2 a_0 b_2^2 \lambda + 3k\mu^4 a_0^2 b_2 = 0
$$

(4.5)
Solving the algebraic equation \( s_1 \) in (4.5) by using symbolic computation software like Maple or Mathematica, we obtain the following solutions:

\[ \phi^3: k\mu^4a_1^3 + ka_1^2\lambda^4\sigma^2 + 6\tau a_1\lambda^4\sigma^2 - 3k\mu^2a_1b_1^2\lambda + 6\tau a_1\mu^4 - 3ka_1b_1^2\lambda^3\sigma + 2k\mu^2a_1^2\lambda^2\sigma + 12\tau a_1\mu^2\lambda^2\sigma = 0 \]

\[ \psi: -kb_1^2\lambda^4\sigma + 3\tau b_1\lambda^4\sigma^2 - 3\tau b_1\mu^4 + 6k\mu^3a_0b_1^2\lambda + 3\omega b_1\lambda^4\sigma^2 + 6k\mu^2a_0^2b_1\lambda^2\sigma + 3k\mu^2b_1^3\lambda^2\sigma + 6\omega b_1\mu^2\lambda^2\sigma + 3ka_0^3b_1^2\lambda^3\sigma + 3\omega b_1\mu^4 + 3k\mu^4a_0^2b_1 = 0 \]

\[ \phi\psi: -9\tau a_1\mu\lambda^4\sigma^2 + 6k\mu^4a_0a_1b_1 + 6k\mu a_1b_1^2\lambda^3\sigma + 12k\mu^2a_0a_1b_1\lambda^2\sigma + 6ka_0a_1b_1\lambda^4\sigma^2 + 6k\mu^3a_1b_1^2\lambda - 9\tau a_1\mu^5 - 18\tau a_1\mu^3\lambda^2\sigma = 0 \]

\[ \phi^2\psi: 12\tau b_1\lambda^2\sigma\mu^2 + 3ka_1^2b_1\lambda^4\sigma^2 - k\mu^2b_1^3\lambda + 3k\mu^4a_1^2b_1 + 6k\mu^2a_1^2b_1\lambda^2\sigma + 6\tau b_1\lambda^4\sigma^2 + 6\tau b_1\mu^4 - k b_1^3\lambda^3\sigma = 0 \]

Solving the algebraic equations in (4.5) by using symbolic computation software like Maple or Mathematica, we obtain the following solutions:

\[ a_0 = 0, a_1 = \pm \sqrt{-\frac{3\tau}{2k}}, b_1 = \pm \sqrt{-\frac{3\tau(\lambda^2\sigma + \mu^2)}{2k\lambda}} \text{ and } \omega = -\frac{\tau \lambda}{2}. \]

Substituting these values into (4.4), we obtain the solution of the mKdV equation (4.2) in the following form:

\[ u_{11}(\xi) = \pm \sqrt{-\frac{3\tau}{2k}} \times \frac{A_1 [\sqrt{-\lambda + \kappa} \cosh(\sqrt{-\lambda + \kappa} \xi) + A_2 \sqrt{-\lambda + \kappa} \sinh(\sqrt{-\lambda + \kappa} \xi)]}{A_1 \sinh(\sqrt{-\lambda + \kappa} \xi) + A_2 \cosh(\sqrt{-\lambda + \kappa} \xi) + \frac{\kappa}{\lambda}} \]

\[ \pm \sqrt{-\frac{3\tau(\lambda^2\sigma + \mu^2)}{2k\lambda}} \times \frac{1}{A_1 \sinh(\sqrt{-\lambda + \kappa} \xi) + A_2 \cosh(\sqrt{-\lambda + \kappa} \xi) + \frac{\kappa}{\lambda}}, \quad (4.6) \]

where \( \sigma = A_1^2 - A_2^2 \).

Since \( A_1 \) and \( A_2 \) are integral constants, one might arbitrarily choose their values. Therefore, if we choose \( A_1 = 0, A_2 \neq 0 \) and \( \mu = 0 \) in (4.6), we obtain the solitary wave solution:

\[ u_{12} = \pm \sqrt{-\frac{3\tau}{2k}} \times \tanh(\sqrt{-\lambda} \xi) \pm \sqrt{\frac{3\tau(\lambda^2\sigma + \mu^2)}{2k\lambda}} \times \text{sech}(\sqrt{-\lambda} \xi), \quad (4.7) \]
Again if we choose $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in (4.6), we find the solitary wave solution

$$u_{13} = \pm \sqrt{-\frac{3\pi}{2k}} \sqrt{-\lambda} \times \coth\left(\sqrt{-\lambda} \xi\right) \pm \sqrt{\frac{3\pi(\lambda^2 + \mu^2)}{2k \lambda}} \times \text{coth}\left(\sqrt{-\lambda} \xi\right)$$

(4.8)

Case 2: In a similar fashion, when $\lambda > 0$, setting equation (4.4) into (4.3) along with equation (3.3) and (3.7) yield a set of algebraic equations for $a_0, a_1, b_1$, and $\omega$ and solving these equations, we attain the following values:

$$a_0 = 0, a_1 = \pm \sqrt{-\frac{3\pi}{2k}}, \quad b_1 = \pm \sqrt{\frac{3\pi(\mu^2 - \lambda^2 \sigma)}{2k \lambda}}, \quad \text{and} \quad \omega = -\frac{\tau A}{2}.$$

We substitute these values into equation (4.4), we obtain the following solution of equation (1.1):

$$u_{14}(\xi) = \pm \sqrt{-\frac{3\pi}{2k}} \times \frac{A_1 \sin\left(\sqrt{\lambda} \xi\right) - A_2 \sin\left(\sqrt{\lambda} \xi\right)}{A_1 \sin\left(\sqrt{\lambda} \xi\right) + A_2 \cos\left(\sqrt{\lambda} \xi\right)} \pm \frac{1}{\frac{A_1 \sin\left(\sqrt{\lambda} \xi\right) + A_2 \cos\left(\sqrt{\lambda} \xi\right)}{2k}}$$

(4.9)

where $\sigma = A_1^2 + A_2^2$.

If we take $A_1 = 0, A_2 \neq 0$ and $\mu = 0$ in (4.9), we attain the solitary wave solution

$$u_{15} = \pm \sqrt{-\frac{3\pi}{2k}} \sqrt{\lambda} \times \tanh\left(\sqrt{\lambda} \xi\right) \pm \sqrt{\frac{3\pi(\lambda^2 - \mu^2 \sigma)}{2k \lambda}} \times \text{sech}\left(\sqrt{\lambda} \xi\right)$$

(4.10)

Again when $A_1 \neq 0, A_2 = 0$ and $\mu = 0$ in (4.6), we get the solitary wave solution

$$u_{16} = \pm \sqrt{-\frac{3\pi}{2k}} \sqrt{\lambda} \times \coth\left(\sqrt{\lambda} \xi\right) \pm \sqrt{\frac{3\pi(\lambda^2 - \mu^2)}{2k \lambda}} \times \text{coth}\left(\sqrt{\lambda} \xi\right)$$

(4.11)

Case 3: In the similar manner, when $\lambda = 0$, by means of equation (4.4) and (4.3) along with equation (3.3) and (3.9), we will find a cluster of algebraic equations, whose solutions are as follows:

$$a_0 = 0, a_1 = \pm \sqrt{-\frac{3\pi}{2k}}, \quad b_1 = \pm \sqrt{\frac{6\mu A_2 \tau - 3\lambda A_1^2}{2k}}, \quad \text{and} \quad \omega = 0.$$

Substituting these values into equation (4.4), we obtain the solution of equation (4.2)

$$u_{17}(\xi) = \pm \sqrt{-\frac{3\pi}{2k}} \times \frac{\mu A_2 \tau + A_1}{2k} \pm \sqrt{\frac{6\mu A_2 \tau - 3\lambda A_1^2}{2k}} \times \frac{1}{\sqrt{\frac{\mu^2 + A_1^2 \xi + A_2^2}{2}} - A_1 \xi + A_2}$$

(4.12)
It is remarkable to see that the traveling wave solution $u_{11}$, $u_{13}$, $u_{14}$, $u_{15}$, and $u_{17}$ of the space time fractional modified KdV equation are fresh and further general and have not been established in the earlier works. These solutions forces to be convenient to investigate the unidirectional and bidirectional model of waves on shallow water surface, long internal wave in a density-stratified ocean and ion acoustic waves in plasma.

4.2: In this subsection, we examine the general and some fresh solutions to the space time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation (1.2) through the double($G'/G$, $1/G$)-expansion method. This equation is significant in the physical phenomena and it involve the Fisher equation, Huxlay equation, Burgers equation, Chaffee-Infanfe equation and Fitzhugh-Nagumo equation. Feng et al. [XIII] have done with this equation using the ($G'/G$)-expansion method when $\alpha = 1$. After that Gepreel [XIV] investigate this equation by homotopy perturbation method. Recently, Hariharan [XIX] develop the analytical approximate solution of this equation by homotopy analysis method and made a comparison with other methods.

For the space time fractional KPP equation, we introduce the following transformation:

$$\xi = \frac{ka^\alpha}{\Gamma(1+\alpha)} + \frac{c t^\alpha}{\Gamma(1+\alpha)} u(x, t) = u(\xi),$$  \hspace{1cm} (4.13)

where $c$ is the velocity of the traveling wave. Using traveling wave variable (4.13) equation (1.2) reduces to the following ODE for $u = u(\xi)$:

$$cu' - k^2 u'' + \mu_1 u + \gamma u^2 + \delta u^3 = 0.$$  \hspace{1cm} (4.14)

Balancing the highest order derivatives and nonlinear terms, we obtain $N = 1$. Therefore, the solution of equation (3.12) is reduces to the form:

$$u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi).$$  \hspace{1cm} (4.15)

Case1: For $\lambda < 0$, inserting equation (4.15) into equation (4.14) along with equation (3.3) and equation (3.5) yields a set of algebraic equations as follows:

$$\phi^0: 3a_0 b_1^2 \lambda^2 \mu^2 + 2ca_1 \lambda^3 \sigma^2 - 2\delta a_2 \lambda \sigma^2 + k^2 b_1 \lambda \mu c + 3\delta a_2 b_2 \lambda \sigma - 2\mu a_0 d \sigma^2 - 2\gamma a_2 b_1 \sigma^2 - \gamma a_2 b_1 \sigma^2 - \mu_1 a_1 \sigma^2 - \delta a_0 b_3 \lambda^4 \sigma^2 + k^2 b_1 \lambda^2 \mu^2 + ca_1 \lambda^2 \sigma^2 + ca_1 \lambda^2 \mu^2 - 2d b_2 \lambda^2 \mu + \gamma b_1 \lambda^2 \sigma + \gamma b_1 \lambda^2 \mu^2 - \mu_1 a_0 \lambda^4 - \delta a_0 \lambda^4 - \gamma a_0 \lambda^4 = 0.$$
Solving the algebraic equation in (4.16) with the help of the symbolic computation software like Maple or Mathematica, we achieve the following solutions:

Set 1: \( a_0 = -\frac{\gamma}{2\delta} \), \( a_1 = \pm \frac{1}{2} \frac{\sqrt{-\lambda(-4\mu_1^2 \delta + \gamma^2)}}{\lambda \delta} \), \( b_1 = \pm \frac{1}{2} \frac{1}{\sqrt{(\gamma^2 \mu^2 - 4\mu_1^2 \mu^2 \delta + \gamma^2 \mu^2 \delta + 4\mu_1^2 \lambda^2 \sigma \delta - 4\mu_1^2 \lambda^2 \sigma \delta)^2 + \lambda \delta^2}} c = \pm \frac{1}{2} \frac{\sqrt{-\lambda(-4\mu_1^2 \delta + \gamma^2)}}{\lambda \delta} \), and \( k = \pm \frac{1}{2} \frac{\sqrt{-2\lambda \delta (-4 \mu_1^2 \delta + \gamma^2)}}{\lambda \delta} \).
Now we attain the following exact solution of the space time fractional KPP equation (1.2) for set 1:

\[ u_{21}(\xi) = -\frac{y}{2\delta} \pm \frac{1}{2} \sqrt{\frac{(-4\mu_1\delta + y^2)}{\delta}} \times \tanh(\sqrt{-\lambda}\xi) \]

\[ \pm \frac{1}{2} \sqrt{\frac{y^2\sigma - 4\mu_2^2\lambda^2\sigma^2}{\lambda\delta}} \times \text{sech}(\sqrt{-\lambda}\xi) \]  

(4.18)

where \( \sigma = A_1^2 - A_2^2 \).

Here \( A_1 \) and \( A_2 \) of constants of integration. Therefore, somebody can freely select their values. If we select \( A_1 = 0, A_2 \neq 0 \) and \( \mu = 0 \) in (4.13), we obtain the solitary wave solution

\[ u_{22}(\xi) = \frac{-y}{2\delta} \pm \frac{1}{2} \sqrt{\frac{(-4\mu_1\delta + y^2)}{\delta}} \times \tanh(\sqrt{-\lambda}\xi) \]

\[ \pm \frac{1}{2} \sqrt{\frac{y^2\sigma - 4\mu_2^2\lambda^2\sigma^2}{\lambda\delta}} \times \text{sech}(\sqrt{-\lambda}\xi) \]  

(4.17)

Again if we select \( A_1 \neq 0, A_2 = 0 \) and \( \mu = 0 \) in (4.13), we attain the solitary wave solution

\[ u_{23}(\xi) = \frac{-y}{2\delta} \pm \frac{1}{2} \sqrt{\frac{(-4\mu_1\delta + y^2)}{\delta}} \times \coth(\sqrt{-\lambda}\xi) \]
Case 2: In the similar way, when \( \lambda > 0 \) setting equation (4.15) into (4.14) along with equation (3.3) and (3.7) yields a set of algebraic equation for \( a_0, a_1, b_1, c, k \) and solving these equation we attain the following solutions:

Set 1:

\[
\begin{align*}
    a_0 &= \frac{-y}{2\delta}, \\
    a_1 &= \pm \frac{1}{2} \sqrt{-\lambda(-4\mu_1 \delta+y^2)} \\
    b_1 &= \pm \frac{1}{2} \sqrt{(y^2-4\mu_1 \delta-\lambda^2 \sigma+4\mu_2 \lambda \delta \sigma)}, \\
    c &= \pm \frac{1}{2} \sqrt{-\lambda(-4\mu_1 \delta+y^2)} \\
    k &= \pm \frac{1}{2} \sqrt{2\lambda \delta(-4\mu_1 \delta+y^2)}.
\end{align*}
\]

and

\[
\begin{align*}
    \alpha_0 &= \pm \frac{1}{4} \sqrt{\frac{y^2-4\mu_1 \delta}{\delta}}, \\
    \alpha_1 &= \pm \frac{1}{2} \sqrt{-\frac{\lambda \delta y(-\frac{1}{2}\frac{y^2-4\mu_1 \delta}{\delta})+\mu_1}{\lambda \delta y(-\frac{1}{2}\frac{y^2-4\mu_1 \delta}{\delta})+\mu_1}}.
\end{align*}
\]

Set 2:

\[
\begin{align*}
    a_0 &= \pm \frac{1}{4} \sqrt{\frac{y^2-4\mu_1 \delta}{\delta}}, \\
    a_1 &= \pm \frac{1}{2} \sqrt{-\frac{\lambda \delta y(-\frac{1}{2}\frac{y^2-4\mu_1 \delta}{\delta})+\mu_1}{\lambda \delta y(-\frac{1}{2}\frac{y^2-4\mu_1 \delta}{\delta})+\mu_1}}.
\end{align*}
\]

Now, we achieve the following exact solution of the space time fractional KPP equation (1.2) for set 1:

\[
u_{24}(\xi) = -\frac{y}{2\delta} \pm \frac{1}{2} \sqrt{-\lambda(-4\mu_1 \delta+y^2)} \times \frac{A_1 \sqrt{\lambda \delta \cos(\sqrt{\lambda \delta \xi})-A_2 \sin(\sqrt{\lambda \delta \xi})}}{A_1 \sin(\sqrt{\lambda \delta \xi})+A_2 \cos(\sqrt{\lambda \delta \xi})+\frac{\lambda}{\delta}} \\
\pm \frac{1}{2} \sqrt{(y^2-4\mu_1 \delta-\lambda^2 \sigma+4\mu_2 \lambda \delta \sigma)} \times \frac{1}{A_1 \sin(\sqrt{\lambda \delta \xi})+A_2 \cos(\sqrt{\lambda \delta \xi})+\frac{\lambda}{\delta}},
\]

where \( \sigma = A_1^2 + A_2^2 \).

As \( A_1 \) and \( A_2 \) are integral constants, if we set \( A_1 = 0, A_2 \neq 0 \) and \( \mu = 0 \) into (4.20), we obtain the solitary wave solution
Again if we set \( A_1 \neq 0, A_2 = 0 \) and \( \mu = 0 \) in (4.20), we find the solitary wave solution

\[
u_{25}(\xi) = -\frac{\mu}{2} \left[ \frac{1}{2} s \left( -\xi + \sqrt{\xi^2 + 4\mu_1^2 s^2} \right) \right] \times \tanh \left( \sqrt{\xi - \xi^2} \right) + \frac{1}{2} \left[ \frac{1}{2} s \left( -\xi + \sqrt{\xi^2 + 4\mu_1^2 s^2} \right) \right] \times \coth \left( \sqrt{\xi - \xi^2} \right) \times \sech \left( \sqrt{\xi - \xi^2} \right). \tag{4.21}
\]

Case 3: Finally when \( \lambda = 0 \), substituting equation (4.15) into (4.14) along with equation (3.3) and (3.9) yields a set of algebraic equations for \( a_0, a_1, b_1, c, k \) and whose solution are as follows:

\[
a_0 = a_0, a_1 = \pm \sqrt{2\mu A_2}, \quad b_1 = 0, \quad c = 0 \quad \text{and} \quad k = 0
\]

Substituting these values into equation (4.11), we obtain the rational function solution of the space time fractional KPP equation (1.2) as follows:

\[
u_{27}(\xi) = a_0 \pm \sqrt{2\mu A_2} \times \frac{\mu \xi + A_1}{\xi^2 + A_1 \xi + A_2} \tag{4.22}
\]

It is substantial to understand that the traveling wave solution \( \nu_{21}, \nu_{22}, \nu_{23}, \nu_{24}, \nu_{25}, \nu_{26} \) and \( \nu_{27} \) of the fractional KPP equation are all new and very much important which were not originate in the prior studies. This diffusion equation is significant in various physical phenomena. It has superior importance in science and engineering that creates an outstanding model for many systems in various fields.

It is noteworthy to observe that for the values of the constants provided in set 2 (both in case 1 and case 2), we attain much new and general solitary wave solutions which might be useful to analyze the diffusion phenomena. But for simplicity the solutions are omitted here.

V. Conclusion

In this article, we find some new and more general solitary wave solutions of two nonlinear space time fractional differential equation, namely, the time fractional KdV equation and the fractional KPP equation in terms of hyperbolic, trigonometric and rational function solution containing parameters. The achieved solutions of these equations is capable to analyze the mathematical model of waves on shallow water surface, ion acoustic waves in plasma, long internal wave in a density-stratified ocean, acoustic waves on a crystal lattice etc. The competence of the two variables \( (G'/G, 1/G) \)-expansion method is consistent and more general than other methods. We confident that the proposed method might be valuable tool in further research.
Reference


